H

millennia institute

CANDIDATE NAME

CLASS

ADMISSION NUMBER

2019 Preliminary Exams
Pre-University 3

MATHEMATICS **9758/01**

Paper 1 **3 September 2019**

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your admission number, name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Qn No.	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	*	Total
Score													
Max Score	6	6	6	8	8	9	10	10	12	12	13		100

This document consists of **24** printed pages.

[Turn over

1 The nth term of a sequence $u_1, u_2, u_3, \ldots$ is given by $u_n = n^2 + \dfrac{1}{n!}$.

(i) Show that $u_n - u_{n-1} = 2n - 1 + \dfrac{1-n}{n!}$. [2]

(ii) Hence find $\displaystyle\sum_{r=2}^{2n}\left(2r-1+\frac{1-r}{r!}\right)$. [3]

(iii) State, with a reason, if the series in part **(ii)** converges. [1]

2 It is given that

$$\mathrm{f}(x) = \begin{cases} 3-x & \text{for } 0 < x \le 2, \\ \dfrac{1}{6}\left(x^2+2\right) & \text{for } 2 < x \le 4, \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x+4)$ for all real values of x.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-3 \le x \le 10$. [3]

(ii) Find the volume of revolution when the region bounded by the graph of $y = \mathrm{f}(x)$, the lines $x = -1$, $x = 2$ and the x-axis is rotated completely about the x-axis. [3]

3 **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos x^2\right)$. [2]

(ii) Hence find $\displaystyle\int x^3 \sin x^2 \, \mathrm{d}x$. [4]

4 **(a)** The curve C has equation $y = \mathrm{f}(x)$.

(i) Given that $\mathrm{f}(x) = \dfrac{x^2+3x+4}{x+1}$, sketch the curve C, showing the equations of the asymptotes and the coordinates of any turning points and any points of intersection with the axes. [3]

(ii) Hence, state the range of values of x where $\mathrm{f}'(x) > 0$. [2]

(b) The curve with equation $y = \mathrm{g}(x)$ is transformed by a stretch with scale factor 2 parallel to the x-axis, followed by a translation of 1 unit in the negative x-direction and followed by a translation of 1 unit in the positive y-direction. The resulting curve has equation $y = \dfrac{x^2+3x+4}{x+1}$. Find $\mathrm{g}(x)$. [3]

5 **(i)** On the same axes, sketch the graphs of $y=\frac{1}{|x-a|}$ and $y=-b(x-a)$, where a and b are positive constants and $ab>\frac{1}{a}$, stating clearly any axial intercepts and equations of any asymptotes. [3]

(ii) Given that the solution to the inequality $\frac{1}{|x-a|}>-b(x-a)$ is $\frac{1}{2}<x<1$ or $x>1$, find the values of a and b. [4]

(iii) Using the values of a and b found in part **(ii)**, write down the solution to the inequality $\frac{1}{x-a}>-b(x-a)$. [1]

6 **(i)** Given that $\mathrm{f}(x)=\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ where a is a constant, find $\mathrm{f}(0)$, $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$ in terms of a. Hence write down the first two non-zero terms in the Maclaurin series for $\mathrm{f}(x)$. Give the coefficients in terms of e. [5]

(ii) The first two non-zero terms in the Maclaurin series for $\mathrm{f}(x)$ are equal to the first two non-zero terms in the series expansion of $\frac{1}{\sqrt{b+x^2}}$, where b is a constant. By using appropriate expansions from the List of Formulae (MF26), find the possible values of a and b in terms of e. [4]

7 **(a)** Find the exact value of $\int_0^1 \frac{x}{2-x^2}\,\mathrm{d}x$. [3]

(b) The expression $\frac{x^2}{(2-x)^2}$ can be written in the form $A+\frac{B}{2-x}+\frac{C}{(2-x)^2}$.

(i) Find the values of A, B and C. [3]

(ii) Show that $\int_0^1 \frac{x^2}{(2-x)^2}\,\mathrm{d}x=p+q\ln 2$, where p and q are constants to be found. [4]

8 **(a)** Referred to the origin O, the point Q has position vector $\mathbf{q}$ such that

$$\mathbf{q} = 2\mathbf{i} - \frac{3}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}.$$

(i) Find the acute angle between $\mathbf{q}$ and the y-axis. [2]

It is given that a vector $\mathbf{m}$ is perpendicular to the xy-plane and its magnitude is 1.

(ii) With reference to the xy-plane, explain the geometrical meaning of $|\mathbf{q} \cdot \mathbf{m}|$ and state its value. [2]

(b) Referred to the origin O, the point R has position vector $\mathbf{r}$ given by $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where λ is a positive constant and $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. It is known that $\mathbf{c}$ is a non-zero vector that is not parallel to $\mathbf{a}$ or $\mathbf{b}$. Given that $\mathbf{c} \times \mathbf{a} = \lambda\mathbf{b} \times \mathbf{c}$, show that $\mathbf{r}$ is parallel to $\mathbf{c}$. [2]

It is also given that $\mathbf{a}$ is a unit vector that is perpendicular to $\mathbf{b}$ and $|\mathbf{b}| = 2$.

By considering $\mathbf{r} \cdot \mathbf{r}$, show that $|\mathbf{c}| = k\sqrt{4\lambda^2 + 1}$, where k is a non-zero constant. [4]

9 The function f is defined by $\mathrm{f} : x \mapsto 1 + 2\mathrm{e}^{-x^2}$, $x \in \mathbb{R}$.

(i) Show that f does not have an inverse. [2]

(ii) The domain of f is further restricted to $x \leq k$, state the largest value of k for which the function f^{-1} exists. [1]

In the rest of the question, the domain of f is $x \in \mathbb{R}$, $x \leq k$, with the value of k found in part **(ii)**.

(iii) Find $\mathrm{f}^{-1}(x)$. [3]

The function g has an inverse such that the range of g^{-1} is given by $(1, 3]$.

(iv) Explain clearly why the composite function gf exists. [2]

It is given that the composite function gf is defined by $\mathrm{gf}(x) = x$.

(v) State the domain and range of gf. [2]

(vi) By considering $\mathrm{gf}(-2)$, find the exact value of $\mathrm{g}^{-1}(-2)$. [2]

10 As a raindrop falls due to gravity, its mass decreases with time due to evaporation. The rate of change of the mass of a raindrop, m grams, with respect to time, t seconds, is a constant c.

(i) **(a)** Write down a differential equation relating m, t and c. State, with a reason, whether c is a positive or negative constant. [2]

(b) Initially the mass of a raindrop is 0.05 grams and, after a further 60 s, the mass of the raindrop is 0.004 grams. Find m in terms of t. [3]

In recent years, scientists are looking for alternative sources of sustainable energy to meet our energy needs. One approach aims to extract kinetic energy from rain to harvest energy. In order to test for the viability of this approach, scientists need to find the maximum kinetic energy that can be harvested from a falling raindrop.

It is known that the kinetic energy K, in millijoules, of an object falling from rest is given by

$$K = \frac{mg^2t^2}{2},$$

where m grams is the mass of the object at time t seconds and g is a positive constant known as the gravitational acceleration.

In the rest of the question, use your answer in part **(i)(b)** as the mass of a raindrop at time t.

(ii) **(a)** Show by differentiation that the maximum value of the kinetic energy of a raindrop falling from rest is pg^2, where p is a constant to be found . Give your answer correct to 2 decimal places. [5]

(b) It is given that $g = 10$. The kinetic energy of a raindrop falling from rest is 1 millijoules when it hits the surface of the ground. Given that it takes more than one minute for a raindrop to fall from rest to the surface of the ground, find the time taken for a raindrop to hit the surface of the ground. [2]

11 In this question, the distance is measured in metres and time is in seconds.

A radio-controlled airplane takes off from ground level and is assumed to be travelling at a steady speed in a straight line. The position vector of the airplane t seconds after it takes off is given by $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a}$ refers to the position vector of the point where it departs and $\mathbf{b}$ is known as its velocity vector.

(i) Given that the airplane reaches the point P with coordinates $(-9,\ 4,\ 6)$ after 6 seconds and its velocity vector is $-2\mathbf{i}+\mathbf{j}+\mathbf{k}$, find the coordinates of the point where it departs. [1]

Paul stands at the point C with coordinates $(-7,\ 5,\ 2)$ to observe the airplane.

(ii) Find the shortest distance from Paul's position to the flight path of the airplane. [3]

While the airplane is in the air, a drone is seen flying at a steady speed in a straight line with equation

$$\frac{2x-1}{-6}=\frac{y+7}{4}=\frac{z-10}{k}.$$

(iii) Show that the equation of the flight path of the drone can be written as $\mathbf{r} = \mathbf{p} + \lambda\mathbf{q}$, where λ is a non-negative constant and $\mathbf{p}$ and $\mathbf{q}$ are vectors to be determined, leaving your answer in terms of k. [1]

(iv) Given that the flight paths of the radio-controlled airplane and the drone intersect, find k. [3]

At P, the airplane suddenly changes its speed and direction. The position vector of the airplane s seconds after it leaves P is given by

$$\mathbf{r}=\begin{pmatrix}-9\\4\\6\end{pmatrix}+s\begin{pmatrix}3\\2\\-1\end{pmatrix},\ \text{where } s\in\mathbb{R},\ s\geq 0.$$

It travels at a steady speed in a straight line towards an inclined slope, which is assumed to be a plane with equation

$$x-y+7z=2.$$

(v) Determine if the new flight path is perpendicular to the inclined slope. [2]

(vi) The airplane eventually collides with the slope. Find the coordinates of the point of collision. [3]

End of Paper