Temasek Junior College
2020 JC 2 Prelim Exam H2 Mathematics Paper 1

1 Given that $A=\sqrt{1+\frac{1}{x^2}}$, and that the **rate of decrease** of A is k times the rate of increase of x when $x = k$, find k, where k is a positive real number. [4]

Solution	Comment for students
Method 1 $A=\sqrt{1+\frac{1}{x^2}}$ $\frac{dA}{dx}=\frac{1}{2}\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}}\left(-\frac{2}{x^3}\right)$ **Chain Rule** $=-\frac{1}{x^3\sqrt{1+\frac{1}{x^2}}}$ ----- (1) When $x = k$, $\frac{dA}{dt}=-k\frac{dx}{dt}$ --- (2) Subst (1) and (2) into $\frac{dA}{dt}=\frac{dA}{dx}\times\frac{dx}{dt}$: $-k\frac{dx}{dt}=-\frac{1}{k^3\sqrt{1+\frac{1}{k^2}}}\frac{dx}{dt}$ $k^4\sqrt{1+\frac{1}{k^2}}=1$ or $k^3\sqrt{k^2+1}=1$ or $k^8+k^6=1$ From GC, since $k > 0$, $k = 0.905$ (3 s.f.)	**Interpretation of Question:** Students must interpret the given statement correctly: Rate of decrease of A $\frac{dA}{dt}=-k\frac{dx}{dt}$ **Some Common Mistakes to take note:** $2x^{-3}\neq\frac{1}{2x^3}$ **(Negative 3 is only the power of x, not 2)** $\frac{dA}{dx}=\frac{1}{2}\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}}\left(-2x^{-1}\right)$ **This is WRONG, differentiation Not integration!** **Use GC to solve equation such as** $k^4\sqrt{1+\frac{1}{k^2}}=1$ or $k^3\sqrt{k^2+1}=1$? Note that if $k^3\sqrt{k^2+1}=1$ $k^3\neq1$, $\sqrt{k^2+1}\neq1$. **Cannot equate to RHS which is non-zero.**
Method 2 $A=\sqrt{1+\frac{1}{x^2}}\Rightarrow A^2=1+\frac{1}{x^2}$ Differentiate wrt t, $2A\frac{dA}{dt}=-\frac{2}{x^3}\frac{dx}{dt}$ --- (1) Subst $x=k$, $\frac{dA}{dt}=-k\frac{dx}{dt}$ into (1):	**Same comment as above.**

$2\sqrt{1+\frac{1}{k^2}}\left(-k\frac{\mathrm{d}x}{\mathrm{d}t}\right)=-\frac{2}{k^3}\frac{\mathrm{d}x}{\mathrm{d}t}$ $k^4\sqrt{1+\frac{1}{k^2}}=1$ or $k^3\sqrt{k^2+1}=1$ From GC, since $k>0$, $k=0.905$ (3 s.f.)	

2 **(a)** The region R is bounded by the curve $y = xe^{-x}$, the line $x = n$ and the x-axis, where n is a positive real number.

(i) Find, in terms of n, the volume of the solid generated, V, when R is rotated 2π radians about the x-axis. [3]

(ii) Hence find $\lim_{n\to\infty} V$, giving your answer in exact form. [1]

[You may assume that as $n \to \infty$, $ne^{-2n} \to 0$ and $n^2e^{-2n} \to 0$.]

(b) **Describe** a sequence of two transformations such that the graph of $y = a^x$ is mapped onto the graph of $y = a^{b-x}$, where a and b are real numbers. [2]

[Solution]

(a)(i) Volume $= \pi\int_0^n \left(xe^{-x}\right)^2 dx = \pi\int_0^n x^2e^{-2x}\, dx$

$$= \pi\left(\left[x^2\left(\frac{e^{-2x}}{-2}\right)\right]_0^n - \int_0^n \left(\frac{e^{-2x}}{-2}\right)(2x)\, dx\right)$$

$$= \pi\left(-\frac{1}{2}\left[x^2e^{-2x}\right]_0^n + \int_0^n xe^{-2x}\, dx\right)$$

$$= \pi\left(-\frac{1}{2}\left[n^2e^{-2n}\right] + \left(-\frac{1}{2}\left[xe^{-2x}\right]_0^n + \frac{1}{2}\int_0^n e^{-2x}\, dx\right)\right)$$

$$= \pi\left(-\frac{1}{2}n^2e^{-2n} - \frac{1}{2}\left[ne^{-2x}\right] - \frac{1}{4}\left[e^{-2x}\right]_0^n\right)$$

$$= \pi\left(-\frac{1}{2}n^2e^{-2n} - \frac{1}{2}ne^{-2n} - \frac{1}{4}\left[e^{-2n} - 1\right]\right)$$

$$= \pi e^{-2n}\left(-\frac{1}{2}n^2 - \frac{1}{2}n - \frac{1}{4}\right) + \frac{\pi}{4}$$

Simplify before performing the next integration by parts

(ii) As $n \to \infty$, $e^{-2n} \to 0$, $ne^{-2n} \to 0$ and $n^2e^{-2n} \to 0$ and so $V \to \frac{\pi}{4}$

V "tends to"

Thus $\lim_{n\to\infty} V = \frac{\pi}{4}$

Limit of V "equals to"

Volume cannot be negative! If you get a negative value, you should correct your answer in **(i)**

(b) $y = a^x \xrightarrow{\text{reflection}} y = a^{-x} \xrightarrow{\text{scaling}} y = a^b a^{-x} = a^{b-x}$

The transformations are (in either order):

1. **Reflection** <u>about</u> y-axis.
2. **Scaling** <u>parallel to</u> y-axis by <u>**factor**</u> a^b.

Use the **correct terminology**. Will not accept "transform", "shift", "flip", "against",etc

OR

$y = a^x \xrightarrow{\text{translation}} y = a^{x+b} \xrightarrow{\text{reflection}} y = a^{(-x)+b}$

1. **Translation** of b <u>units</u> in the <u>negative</u> direction of the x-axis.
2. **Reflection** <u>about</u> y-axis.

OR

$y = a^x \xrightarrow{\text{reflection}} y = a^{-x} \xrightarrow{\text{translation}} y = a^{-(x-b)} = a^{b-x}$

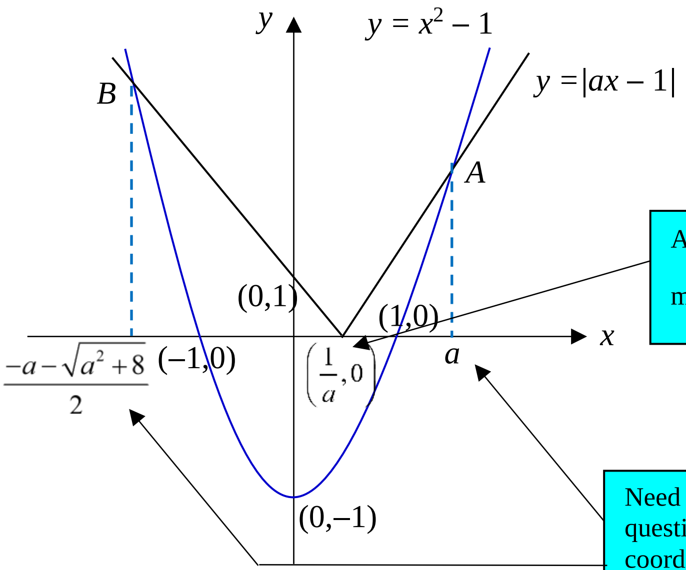
1. **Reflection** <u>about</u> y-axis.
2. **Translation** of b units in the <u>positive</u> direction of the x-axis.

3 On the same diagram, sketch the graphs of $y = |ax - 1|$, where $a > 1$, and $y = x^2 - 1$. Give, in terms of a, the coordinates of the points where the curves meet the axes and the x-coordinates of their intersection points. [4]

(i) Hence find the solution set of the inequality $|ax - 1| > x^2 - 1$. [1]

(ii) Deduce the solution set of the inequality $|a^{x+1} - 1| > a^{2x} - 1$. [2]

[Solution]



As $a > 1$, need to ensure that vertex of $y=|ax-1|$ must be in between of $0 < \frac{1}{a} < 1$

Need to show in the graph as required by the question. There is not necessary to find the y-coordinate of the intersection.

At A: $ax - 1 = x^2 - 1$

$x^2 - ax = 0$

$x(x - a) = 0$

$x = 0$ (rej) or $x = a$

At B: $-(ax - 1) = x^2 - 1$

$x^2 + ax - 2 = 0$

Since $x < 0$, $x = \dfrac{-a - \sqrt{a^2 + 8}}{2}$

There is a need to show working of finding the x-coordinate of the intersection of the 2 curves.

(i) From the graph, the solution set is $\left\{ x : x \in \mathbb{R}, \dfrac{-a - \sqrt{a^2 + 8}}{2} < x < a \right\}$

Need to write the solution in set format (required by the question).

(ii) Replace x by a^x,

$$\frac{-a - \sqrt{a^2 + 8}}{2} < a^x < a$$

There is solution for this inequality and should not be "rejected" or "NA".

Since $\dfrac{-a - \sqrt{a^2 + 8}}{2} < 0$, we cannot use logarithm to solve the inequality.

$\Rightarrow \dfrac{-a - \sqrt{a^2 + 8}}{2} < a^x$ and $a^x < a$

$\Rightarrow x \in \mathbb{R}$ and $x < 1$

$\left(\because \dfrac{-a - \sqrt{a^2 + 8}}{2} < 0 < a^x \text{ for all real } x\right)$

$\Rightarrow x < 1$

The solution set is $\{x : x \in \mathbb{R}, x < 1\}$

4 **(a)** The complex number w is such that $w^3 = -8\text{i}$. Given that one possible value of w is 2i, use a **non-calculator method** to find the other values of w. Give your answers in the form $a + b\text{i}$, where a and b are exact values. [3]

(b) Find the principal argument of the complex number $q = -1 + \text{i}\sqrt{3}$. Hence, or otherwise, show that there is no integer value of n for which the real part of $\dfrac{q^n}{q^*}$ is zero. [4]

[Solution]

(a) **Given 2i is a root of the equation, so $w - 2\text{i}$ is a factor of $w^3 + 8\text{i}$.**
So the other factor is a quadratic term $w^2 + aw + b$ where a and b may not be real numbers (cannot assume they are real numbers) as the original equation has complex coefficients.

(a) $w^3 + 8\text{i} = (w - 2\text{i})(w^2 + aw + b)$, $a,\ b \in \mathbb{C}$

Comparing the constant term : $8\text{i} = -2b\text{i} \Rightarrow b = -4$

Comparing the w^2 term: $0 = -2\text{i} + a \Rightarrow a = 2\text{i}$

Or using the **long division method** to obtain the quadratic factor $w^2 + 2\text{i}\,w - 4$

$w^3 + 8\text{i} = 0 \Rightarrow (w - 2\text{i})(w^2 + 2w\text{i} - 4) = 0$

For $w^2 + 2w\text{i} - 4 = 0$

$$w = \frac{-2\text{i} \pm \sqrt{-4 - 4(-4)}}{2} = \frac{-2\text{i} \pm 2\sqrt{3}}{2} = -\text{i} \pm \sqrt{3}$$

The other values of w are $\sqrt{3} - \text{i}$ or $-\sqrt{3} - \text{i}$

Alternative Solution

Let $w = x + y\text{i}$, $x,\ y \in \mathbb{R}$

$w^3 = -8\text{i} \Rightarrow (x + y\text{i})^3 = -8\text{i}$

$x^3 - 3xy^2 + \text{i}\,(3x^2y - y^3) = 0 - 8\text{i}$

Comparing the real part:

$x^3 - 3xy^2 = 0 \Rightarrow x(x^2 - 3y^2) = 0$

$x = 0$ or $x^2 = 3y^2 \Rightarrow x = \pm\sqrt{3}y$

Comparing the imaginary part:

$3x^2y - y^3 = -8$

Subst $x = 0$, $-y^3 = -8 \Rightarrow y = 2$

Subst $x^2 = 3y^2$, $3(3y^2)y - y^3 = -8$

$\Rightarrow y^3 = -1 \Rightarrow y = -1$

Thus $x = \pm\sqrt{3}y = \pm\sqrt{3}$

The other values of w are $\sqrt{3} - \text{i}$ or $-\sqrt{3} - \text{i}$

(b) $q = -1 + \text{i}\sqrt{3}$ **(second quadrant)**

Basic angle, $\tan\alpha = \sqrt{3} \Rightarrow \alpha = \dfrac{\pi}{3}$

$\therefore \arg q = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

$$\arg\left(\frac{q^n}{q^*}\right) = n\arg q - \arg q^* = n\arg q + \arg q = (n+1)\frac{2\pi}{3}$$

For real part = 0, the complex number $\frac{q^n}{q^*}$ must lie on the imaginary axis

You need to consider the general term as follows:

$$(n+1)\frac{2\pi}{3} = \pm\frac{(2k+1)}{2}\pi \quad \text{where } k \in \mathbb{Z}$$

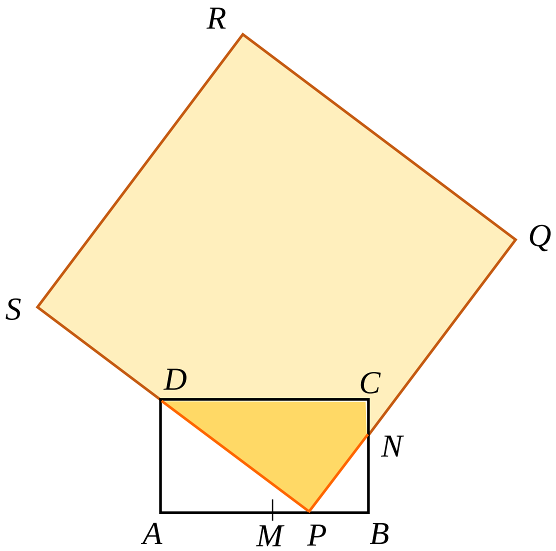
$$n+1 = \pm\frac{3(2k+1)}{4}$$

cannot just list a few angles, say $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ..$

Since $3(2k+1)$ is odd for all $k \in \mathbb{Z}$, $\frac{3(2k+1)}{4}$ is never an integer.

Thus there is no integer value of *n* such that the real part of $\frac{q^n}{q^*}$ is zero.

5



The diagram (not drawn to scale) shows a piece of rectangular paper *ABCD* with *AB* = 10 cm, *BC* = 5 cm and *M* is the midpoint of *AB*. A piece of square translucent cellophane paper *PQRS*, with side 15 cm, is overlaid on *ABCD* such that *P* lies on *MB*, *SDP* is collinear, and *PQ* intersects *BC* at *N*.

Means *P* can lie on *M* or *B* or anywhere in between.

(i) Denoting the length of *AP* by *x* cm, show that $BN = \frac{x(10-x)}{5}$. [2]

(ii) Using an analytical method, find the exact maximum area of the region in *ABCD* overlaid by *PQRS*. [6]

Means use of algebraic method, i.e. no calculator is allowed

[Solution]

(i) As this is a "show" question with the answer given, it is important to explain/state the concepts (e.g. similar triangles) used in your argument and how they lead to the given result.

$\angle APD = \angle BNP$ and $\angle DAP = \angle PBN$

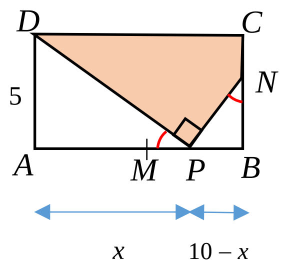
Therefore ΔAPD is similar to ΔBNP (AA)

Spell out "similar" as '~' is not in Cambridge standard notation list.

$$\frac{AD}{AP} = \frac{BP}{BN}$$

$$\frac{5}{x} = \frac{10-x}{BN}$$

$$BN = \frac{x(10-x)}{5} \quad \text{(shown)}$$



Alternative (Using trigonometry)

$\angle APD = \angle BNP$

Therefore $\tan\angle APD = \tan\angle BNP$

$$\frac{AD}{AP} = \frac{BP}{BN}$$

$$\frac{5}{x} = \frac{10-x}{BN}$$

$$BN = \frac{x(10-x)}{5} \quad \text{(shown)}$$

(ii) "**Using an analytical method**" means **using an algebraic method**, i.e. **no calculator is allowed**. Hence differentiation needs to be used and quadratic equations need to be solved by using the quadratic formula or by factorisation and not by GC.

Let E be the area of the region overlaid by $PQRS$.

$E =$ Area of $ABCD$ – Area of DAP – Area of PNB

$$= (5\times10) - \frac{1}{2}(x)(5) - \frac{1}{2}(10-x)\frac{x(10-x)}{5}$$

$$= 50 - \frac{25}{2}x + 2x^2 - \frac{x^3}{10}$$

$$\frac{\mathrm{d}E}{\mathrm{d}x} = -\frac{3}{10}x^2 + 4x - \frac{25}{2} = -\frac{1}{10}(3x^2 - 40x + 125)$$

When $\frac{\mathrm{d}E}{\mathrm{d}x} = 0$, $-\frac{1}{10}(3x^2 - 40x + 125) = 0$

$$-\frac{1}{10}(3x-25)(x-5) = 0$$

$$x = \frac{25}{3} \quad \text{or} \quad x = 5$$

need to show factorization or quadratic formula to solve quadratic equation

Common mistake is to reject $x = 5$ as students wrongly assumed P cannot be B

$$\frac{\mathrm{d}^2E}{\mathrm{d}x^2} = 4 - \frac{3}{5}x$$

When $x = \frac{25}{3}$, $\frac{\mathrm{d}^2E}{\mathrm{d}x^2} = -1 < 0$;

When $x = 5$, $\frac{\mathrm{d}^2E}{\mathrm{d}x^2} = 1 > 0$

Second derivative test (for maximum pt) is strongly encouraged in this question. Actual values must be calculated & shown.

Need to test $x = 5$ to show E is not a maximum at this x value

Hence E is maximum when $x = \frac{25}{3}$

Max area overlaid by $PQRS$

$$= 50 - \frac{25}{2}\left(\frac{25}{3}\right) + 2\left(\frac{25}{3}\right)^2 - \frac{1}{10}\left(\frac{25}{3}\right)^3$$

$$= \frac{725}{27}\text{ cm}^2$$

Exact answer required

Alternative (first derivative test)

Mark is not awarded if no supporting values are given in test. Not encouraged here as it is tedious.

x	5^- (e.g. 4.9)	5	5^+ (e.g. 5.1)		$\frac{25}{3}^-$ (e.g. 8.3)	$\frac{25}{3}$	$\frac{25}{3}^+$ (e.g. 8.4)
$\frac{\mathrm{d}E}{\mathrm{d}x}$	< 0 (e.g. –0.103)	0	> 0 (e.g. 0.097)		> 0 (e.g. 0.033)	0	< 0 (e.g.– 0.068)
	\	—	/		/	—	\

Strongly suggests method of differences

6 **(i)** Given that $T_r = \frac{r^2}{2^r}$, show that $T_r - T_{r+1} = \frac{(r-1)^2}{2^{r+1}} - \frac{1}{2^r}$. [2]

(ii) Hence, find $\sum_{r=1}^{N}\left(\frac{(r-1)^2}{2^{r+1}} - \frac{1}{2^r}\right)$ giving your answer in the form $\frac{1}{2} - \mathrm{f}(N)$. [2]

Notice links between parts

(iii) Show that $\sum_{r=1}^{N}\frac{(r-1)^2}{2^{r+1}} = \frac{3}{2} - \frac{(N+1)^2}{2^{N+1}} - \frac{1}{2^N}$. [2]

(iv) Deduce an expression for $\sum_{r=1}^{N-1}\frac{r^2}{2^r}$ in terms of N. [3]

[Solution]

(i) $T_r - T_{r+1}$

$$= \frac{r^2}{2^r} - \frac{(r+1)^2}{2^{r+1}}$$

$$= \frac{2r^2 - (r^2 + 2r + 1)}{2^{r+1}}$$

The LCM of 2^r and 2^{r+1} is 2^{r+1}

$$= \frac{r^2 - 2r - 1}{2^{r+1}}$$

$$= \frac{(r-1)^2 - 2}{2^{r+1}}$$

"Show" implies that all steps **must be shown**, as the final answer is given.

$$= \frac{(r-1)^2}{2^{r+1}} - \frac{1}{2^r}$$

(ii) $\sum_{r=1}^{N}\left(\frac{(r-1)^2}{2^{r+1}} - \frac{1}{2^r}\right) = \sum_{r=1}^{N}(T_r - T_{r+1})$ using **(i)**

$$\begin{aligned} &= T_1 - \cancel{T_2} \\ &\quad + \cancel{T_2} - \cancel{T_3} \\ &\quad + \cancel{T_3} - \cancel{T_4} \\ &\quad + \ldots \\ &\quad + \cancel{T_{N-1}} - \cancel{T_N} \\ &\quad + \cancel{T_N} - T_{N+1} \\ &= T_1 - T_{N+1} \\ &= \frac{1}{2} - \frac{(N+1)^2}{2^{N+1}} \end{aligned}$$

Presentation:
(1) Always start your solution with the given expression, in this case, $\sum_{r=1}^{N}\left(\frac{(r-1)^2}{2^{r+1}} - \frac{1}{2^r}\right)$. Do not start your solution with "=", for example.
(2) Use brackets for expressions in sigma notations, i.e. $\sum_{r=1}^{N}(T_r - T_{r+1})$ instead of $\sum_{r=1}^{N}T_r - T_{r+1}$.
$\sum_{r=1}^{N}T_r - T_{r+1}$ actually means the sum of T_r from 1 to N taking away T_{r+1}.

(iii) From **(ii)**, $\sum_{r=1}^{N}\left(\frac{(r-1)^2}{2^{r+1}}-\frac{1}{2^r}\right)=\frac{1}{2}-\frac{(N+1)^2}{2^{N+1}}$

$$\sum_{r=1}^{N}\frac{(r-1)^2}{2^{r+1}}=\frac{1}{2}-\frac{(N+1)^2}{2^{N+1}}+\sum_{r=1}^{N}\frac{1}{2^r}$$

You should be able to recognise the GP here.

$$=\frac{1}{2}-\frac{(N+1)^2}{2^{N+1}}+\frac{\frac{1}{2}\left(1-\frac{1}{2^N}\right)}{1-\frac{1}{2}}$$

"Show" implies that all steps **must be shown**, as the final answer is given.

$$=\frac{1}{2}-\frac{(N+1)^2}{2^{N+1}}+\left(1-\frac{1}{2^N}\right)$$

$$=\frac{3}{2}-\frac{(N+1)^2}{2^{N+1}}-\frac{1}{2^N}$$

(iv) Replace r by $r-1$,

$$\sum_{r=1}^{N-1}\frac{r^2}{2^r}=\sum_{r-1=1}^{r-1=N-1}\frac{(r-1)^2}{2^{r-1}}$$

$$=\sum_{r=2}^{N}\frac{(r-1)^2}{2^{r-1}}$$

$$=4\sum_{r=2}^{N}\frac{(r-1)^2}{2^{r+1}}$$

Changing the expression to match the expression in (iii), **so that the given result in (iii) can be used**.
Remember to change **ALL** parts of the expression (lower limit, upper limit, and all the r-terms in the expression).

$$=4\sum_{r=1}^{N}\frac{(r-1)^2}{2^{r+1}}$$ since the first term is zero

The missing first term ($r = 1$) is $\frac{(1-1)^2}{2^{1+1}}=0$, so the sum is the same whether we start from $r = 1$ or $r = 2$.

$$=4\left(\frac{3}{2}-\frac{(N+1)^2}{2^{N+1}}-\frac{1}{2^N}\right)$$ using (iii)

$$=6-\frac{(N+1)^2}{2^{N-1}}-\frac{1}{2^{N-2}}$$

Alternatively, replace r by $r+1$ in **(iii)**,

$$\sum_{r+1=1}^{r+1=N}\frac{r^2}{2^{r+2}}=\frac{3}{2}-\frac{(N+1)^2}{2^{N+1}}-\frac{1}{2^N}$$

$$\frac{1}{4}\sum_{r=0}^{N-1}\frac{r^2}{2^r}=\frac{3}{2}-\frac{(N+1)^2}{2^{N+1}}-\frac{1}{2^N}$$

$$\sum_{r=0}^{N-1}\frac{r^2}{2^r}=4\left(\frac{3}{2}-\frac{(N+1)^2}{2^{N+1}}-\frac{1}{2^N}\right)$$

$$=6-\frac{(N+1)^2}{2^{N-1}}-\frac{1}{2^{N-2}}$$

$$\Rightarrow\sum_{r=1}^{N-1}\frac{r^2}{2^r}=6-\frac{(N+1)^2}{2^{N-1}}-\frac{1}{2^{N-2}}$$ since the first term is zero

7 In the latest simulation game, Speedy Cheetah Family Simulator, cheetahs need to hunt down and eat gazelles to maintain their health and survive in the simulated Savana forest. The game simulates how a cheetah chases a gazelle. The animals' movements are modelled by a series of leaps in the same direction in a straight line. The cheetah's first leap is 6 m and each subsequent leap is shorter than its preceding leap by 10 cm. The gazelle's first leap is 3 m and each subsequent leap is 98% of its preceding leap. Assume both the cheetah and the gazelle start leaping at the same moment and the time taken for each leap is the same.

AP

GP

Their leaps are synchronised; meaning at any time, the number of leaps taken for both animals is the same.

(i) Find, in terms of n, the total distance covered by the gazelle after <u>n leaps</u>. [2]

(ii) Find, in terms of n, the total distance covered by the cheetah after n leaps. If the gazelle is 32 m away from the cheetah at the start of the chase, find the number of leaps taken by the cheetah to catch the gazelle. [4]

(iii) Find the number of leaps the cheetah takes before coming to a complete stop. Deduce the shortest distance (to the nearest tenth of a metre) that the gazelle must be from the cheetah at the start of the chase in order to survive the chase. [4]

Make use of above result(s)

[Solution]

(i) Total distance covered by the gazelle <u>after n leaps</u>

$$= \frac{3(1-0.98^n)}{1-0.98}$$

Number of terms is n

$$= 150(1-0.98^n)$$

(ii) Total distance covered by the cheetah after n leaps

$$= \frac{n}{2}\left[2(6)+(n-1)(-0.1)\right]$$

As each leap is shorter than the previous one, common difference, d is negative

$$= 6n - \frac{1}{20}n(n-1)$$

$$= -\frac{n^2}{20} + \frac{121}{20}n$$

For the cheetah to catch the gazelle,

$$-\frac{n^2}{20} + \frac{121}{20}n - 150(1-0.98^n) \geq 32$$

Make use of (i) and (ii) to show the inequality

From GC,

n	$-\frac{n^2}{20} + \frac{121}{20}n - 182 + 150(0.98^n)$
11	$-1.39 < 0$
12	$1.1075 > 0$

Use table since n is an integer

Therefore, the number of leaps is 12

(iii) Distance covered by cheetah at the nth leap $= 6 + (n-1)(-0.1) = 0$

$$n = \frac{6+0.1}{0.1} = 61$$

The number of leaps taken before cheetah stops = 60

Answer the question. Since cheetah has stopped leaping (distance is 0) at the 61st leap, the number of leaps taken is 60.

Let the distance of the gazelle from the cheetah at the start be d m.

Gazelle will survive the chase if $150(1-0.98^{60})+d > \frac{60}{2}\left[2(6)+59(-0.1)\right]$ Sub n = 60

$$d > 77.63$$

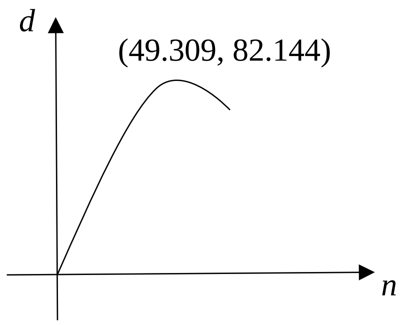
The shortest distance is 77.7 m Round up as the inequality sign is >

A refined solution

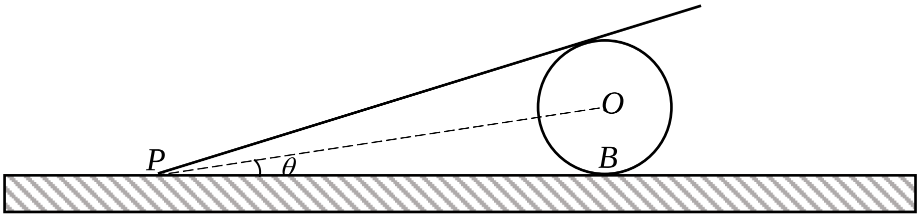
Let $d = \left|150(1-0.98^{n})-\frac{n}{2}\left[2(6)+(n-1)(-0.1)\right]\right|$

From GC, max d = 82.144.

Therefore the shortest distance is 82.2 m



8 The diagram below (not drawn to scale) shows the vertical cross section of a cylindrical drum with centre O and radius 5 cm. The drum is fixed at a point B on the horizontal ground and a rectangular metal sheet leans on it.



The metal sheet is moved in the direction PB so that P is h cm closer to B and $\angle OPB$ is increased by α radians, where α is small.

(i) Initially, the metal sheet touches the horizontal ground at point P such that PB = 50 cm and $\angle OPB = \theta$ radians. Using a suitable compound angle formula in MF26, show that

$$h \approx \frac{505\alpha}{1+10\alpha}.$$ [4]

(ii) Hence express h as a series in α, up to and including the term in α^3. [3]

(iii) Given that P is moving towards B at a rate of 0.1 cm s^{-1}, find the rate of change of α when $h = 1$. [4]

[Solution]

(i) $\tan\theta = \frac{5}{50} = \frac{1}{10}$

$\tan(\theta+\alpha) = \frac{5}{50-h}$

Using compound angle formula

$$\frac{\tan\theta + \tan\alpha}{1-\tan\theta\tan\alpha} = \frac{5}{50-h}$$

Since α is small, $\tan\alpha \approx \alpha$

[It is necessary that you state the above result]

$$\frac{\frac{1}{10}+\alpha}{1-\frac{\alpha}{10}} \approx \frac{5}{50-h}$$

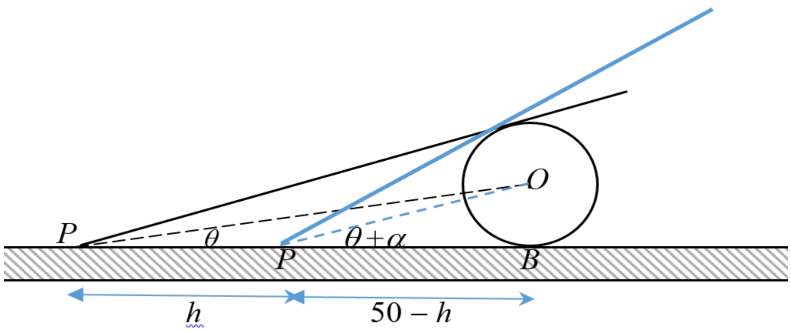
$$\left(\frac{1+10\alpha}{10}\right)(50-h) \approx 5\left(\frac{10-\alpha}{10}\right)$$

$$h \approx 50 - \frac{5(10-\alpha)}{1+10\alpha}$$

Thus $h \approx \frac{505\alpha}{1+10\alpha}$ (shown) --- (1)

"The metal sheet is moved in the direction PB so that P is h cm closer to B and $\angle OPB$ is increased by α radians, where α is small"

Draw a diagram to "understand the above statement.



(ii) $h \approx 505\alpha\left(1+10\alpha\right)^{-1}$

$= 505\alpha\left(1-10\alpha+100\alpha^2+....\right)$

$= 505\left(\alpha-10\alpha^2+100\alpha^3+....\right)$

(iii) $\dfrac{\mathrm{d}h}{\mathrm{d}t} = 505\left(1-20\alpha+300\alpha^2+....\right)\dfrac{\mathrm{d}\alpha}{\mathrm{d}t}$ --- (2)

Subst h = 1 into (1) $\Rightarrow \alpha = \dfrac{1}{495}$

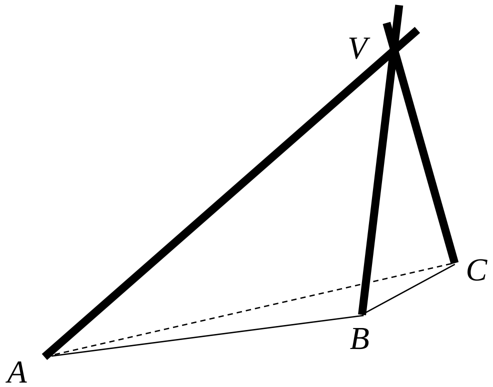
Subst $\dfrac{\mathrm{d}h}{\mathrm{d}t} = 0.1$ into (2), $0.1 = 505\left(1-20\left(\dfrac{1}{495}\right)+300\left(\dfrac{1}{495}\right)^2+....\right)\dfrac{\mathrm{d}\alpha}{\mathrm{d}t}$

$\Rightarrow \dfrac{\mathrm{d}\alpha}{\mathrm{d}t} = 0.000206$ (3sf)

$\dfrac{\mathrm{d}h}{\mathrm{d}t}$ is a positive rate as the distance h is measured from the initial position of P

Thus α increases by 0.000206 radians per second at this instant.

9 A camper built a tripod tent using 3 poles and 2 canvases as shown in the diagram below. The 3 poles rest on **sloping ground** at the points A, B, and C, and are fastened together at the point V. With respect to the origin (not shown in the diagram), A, B and C have position vectors $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $7\mathbf{i} + 4\mathbf{j}$ and $8\mathbf{i} + 12\mathbf{j} + 2\mathbf{k}$ respectively, where **k** is perpendicular to the horizontal. You may assume that the sloping ground is a plane.



(i) Find a vector equation, in **parametric form**, of the sloping ground. [2]

(ii) The vectors $\overrightarrow{AV}$ and $\overrightarrow{CV}$ are parallel to $4\mathbf{i} + \alpha\mathbf{j} + 10\mathbf{k}$ and $-2\mathbf{i} - 2\mathbf{j} + 10\mathbf{k}$ respectively. Find, in either order, the value of α and the position vector of V. [4]

(iii) Find the acute angle between the pole CV and the sloping ground. [3]

(iv) The camper attaches a small torchlight to the top of the tent at V. The torchlight was not fastened properly and dropped vertically down to the ground. Assuming the lowest end of the torch is at V, find the distance travelled by the torchlight from V to the ground. [3]

[Solution]

(i) $\overrightarrow{AB} = \begin{pmatrix}7\\4\\0\end{pmatrix} - \begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}5\\1\\1\end{pmatrix}$ $\qquad \overrightarrow{AC} = \begin{pmatrix}8\\12\\2\end{pmatrix} - \begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}6\\9\\3\end{pmatrix} = 3\begin{pmatrix}2\\3\\1\end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix}8\\12\\2\end{pmatrix} - \begin{pmatrix}7\\4\\0\end{pmatrix} = \begin{pmatrix}1\\8\\2\end{pmatrix}$

Vector equation of **sloping ground**
(or plane ABC)

$\mathbf{r} = \overrightarrow{OA} + s\,\overrightarrow{AB} + t\,\overrightarrow{AC}$ - **parametric form**

$\mathbf{r} = \begin{pmatrix}2\\3\\-1\end{pmatrix} + s\begin{pmatrix}5\\1\\1\end{pmatrix} + t\begin{pmatrix}2\\3\\1\end{pmatrix}, \quad s, t \in \mathbb{R}$

Alternative answers:

$\mathbf{r} = \mathbf{u} + s\mathbf{v} + t\mathbf{w}$

where $\mathbf{u} = \begin{pmatrix}2\\3\\-1\end{pmatrix}$ or $\begin{pmatrix}7\\4\\0\end{pmatrix}$ or $\begin{pmatrix}8\\12\\2\end{pmatrix}$,

v and **w** can be $\overrightarrow{AB}$ or $\overrightarrow{AC}$ or $\overrightarrow{BC}$.

$$\overrightarrow{AV} \neq \begin{pmatrix}4\\ \alpha\\ 10\end{pmatrix} \ \& \ \overrightarrow{CV} \neq \begin{pmatrix}-2\\-2\\10\end{pmatrix}$$

Alternative

Use

$$\overrightarrow{AV} = \lambda\begin{pmatrix}4\\ \alpha\\ 10\end{pmatrix} \Rightarrow \overrightarrow{OV} = \overrightarrow{OA} + \lambda\begin{pmatrix}4\\ \alpha\\ 10\end{pmatrix}, \ \lambda \in \mathbb{R}$$

$$\overrightarrow{CV} = \mu\begin{pmatrix}-2\\-2\\10\end{pmatrix} \Rightarrow \overrightarrow{OV} = \overrightarrow{OC} + \mu\begin{pmatrix}-2\\-2\\10\end{pmatrix}, \ \mu \in \mathbb{R}$$

Equating $\overrightarrow{OV}$: $\overrightarrow{OA} + \lambda\begin{pmatrix}4\\ \alpha\\ 10\end{pmatrix} = \overrightarrow{OC} + \mu\begin{pmatrix}-2\\-2\\10\end{pmatrix}$

(ii) Line AV: $\mathbf{r} = \begin{pmatrix}2\\3\\-1\end{pmatrix} + \lambda\begin{pmatrix}4\\ \alpha\\ 10\end{pmatrix}, \quad \lambda \in \mathbb{R}$

Line CV: $\mathbf{r} = \begin{pmatrix}8\\12\\2\end{pmatrix} + \mu\begin{pmatrix}-2\\-2\\10\end{pmatrix}, \quad \mu \in \mathbb{R}$

Intersection at V:

$$\begin{pmatrix}2\\3\\-1\end{pmatrix} + \lambda\begin{pmatrix}4\\ \alpha\\ 10\end{pmatrix} = \begin{pmatrix}8\\12\\2\end{pmatrix} + \mu\begin{pmatrix}-2\\-2\\10\end{pmatrix}$$

$4\lambda + 2\mu = 6$ --- (1)

$2\mu + \lambda\alpha = 9$ --- (2)

$10\lambda - 10\mu = 3$ --- (3)

Method 1

Solving (1) and (3), $\lambda = \frac{11}{10}$, $\mu = \frac{4}{5}$

Subst into (2), $\alpha = \frac{74}{11}$

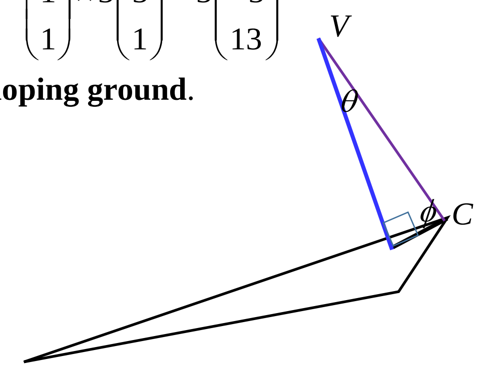
Method 2

From GC: $\lambda = \frac{11}{10}$, $\mu = \frac{4}{5}$, $\lambda\alpha = \frac{37}{5} \Rightarrow \alpha = \frac{74}{11}$

Coordinates of V are $\left(\frac{32}{5}, \frac{52}{5}, 10\right)$

(iii) A normal to the ground (plane ABC) = $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}5\\1\\1\end{pmatrix} \times 3\begin{pmatrix}2\\3\\1\end{pmatrix} = 3\begin{pmatrix}-2\\-3\\13\end{pmatrix}$

Let θ be the **acute angle** between $\boldsymbol{CV}$ and the **sloping ground**.



Define θ properly.

$$\cos\phi = \left|\frac{\begin{pmatrix}-2\\-3\\13\end{pmatrix} \cdot \begin{pmatrix}-2\\-2\\10\end{pmatrix}}{\sqrt{182}\sqrt{108}}\right| = 0.9985745$$

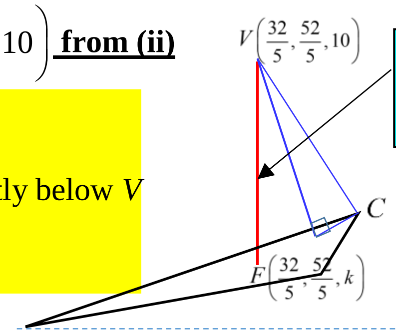
It is important to put modulus to get acute angle.

$\theta = 90° - \phi$

$= 86.9°$

(iv) Let F be the point where the torchlight hits the sloping ground (plane ABC).

Method 1: Using coord of $V\left(\frac{32}{5}, \frac{52}{5}, 10\right)$ from (ii)



The distance required is not perpendicular distance from V to the plane. It is ***VF***.

$\overrightarrow{OF} = \begin{pmatrix} \frac{32}{5} \\ \frac{52}{5} \\ k \end{pmatrix}, \quad k \in \mathbb{R}$ since F is directly below V

Since F lies on the sloping ground,

$$\begin{pmatrix} \frac{32}{5} \\ \frac{52}{5} \\ k \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad s,t \in \mathbb{R}$$

Using GC, $k = \frac{18}{13}$

Therefore the distance travelled = $10 - \frac{18}{13} = \frac{112}{13} = 8.62$ (3 s.f.)

Method 2

Line VF: $\mathbf{r} = \begin{pmatrix} \frac{32}{5} \\ \frac{52}{5} \\ 10 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$

Line VF is not parallel to the normal of the plane ABC. It is parallel to **k**.

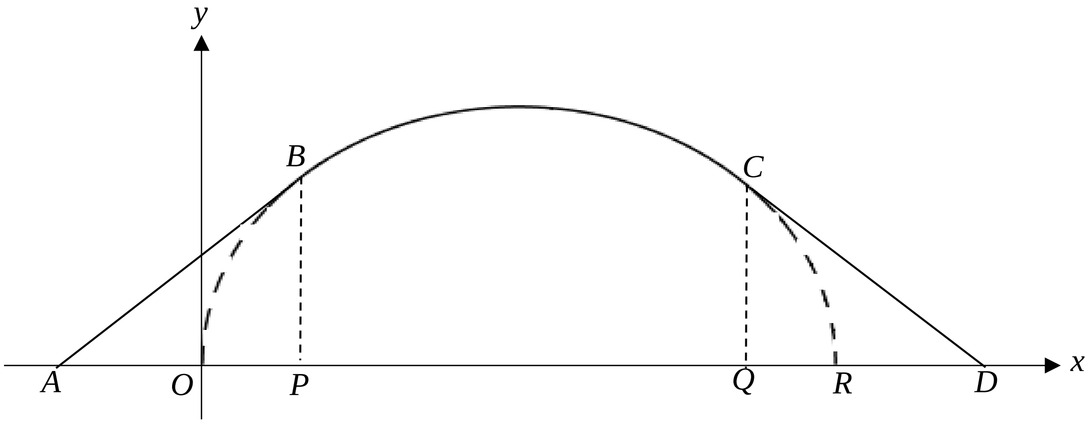
Plane ABC: $\mathbf{r}.\begin{pmatrix} -2 \\ -3 \\ 13 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}.\begin{pmatrix} -2 \\ -3 \\ 13 \end{pmatrix} = -26 \Rightarrow \mathbf{r}.\begin{pmatrix} 2 \\ 3 \\ -13 \end{pmatrix} = 26$

At intersection F, $\left[\begin{pmatrix} \frac{32}{5} \\ \frac{52}{5} \\ 10 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right].\begin{pmatrix} 2 \\ 3 \\ -13 \end{pmatrix} = 26$

$$\Rightarrow t = -\frac{112}{13}$$

Therefore the distance travelled $= \frac{112}{13} = 8.62$ (3 s.f.) (F is directly below V)

10 The figure below shows a **symmetrical** design for a suspension bridge arch *ABCD*.



The arch design consists of the curved part *BC* and the straight lines *AB* and *CD*. The curved part *BC* is part of the curve *OBCR* with parametric equations given by

$$x = a(2t - \sin 2t), \quad y = a(1 - \cos 2t) \text{ for } 0 \le t \le \pi,$$

where *a* is a constant.

(i) Find, in terms of *a*, the length of line segment *OR* and the height of the bridge at its highest point. [3]

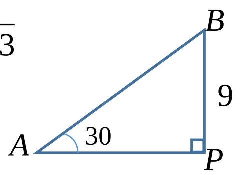
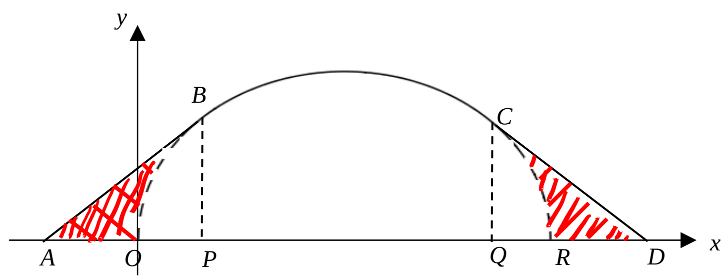
(ii) Show that $\frac{dy}{dx} = \cot t$. [2]

The straight lines *AB* and *CD* are tangents to the curve at *B* and *C* respectively and are inclined at 30 degrees to the horizontal.

(iii) Find the exact coordinates of *B*, in terms of *a*. [2]

(iv) Given *a* = 6, show that the area of the region bounded by the arch *ABCD*, the curve *OBCR* and the *x*-axis is given by $h - k\int_0^{\alpha}(1-\cos 2t)^2 \, dt$, where *h*, *k* and α are constants to be determined. Hence find this exact area. [6]

Solution	**Comment**
(i) At *R*, $y = 0 \Rightarrow t = \pi$ $x = a(2\pi - \sin 2\pi) = 2\pi a$ Length of *OR* = $2\pi a$ $y = a(1 - \cos 2t)$ For $0 \le t \le \pi$, minimum value of cos2*t* is –1. Maximum value of $y = a(1 - (-1)) = 2a$ The height of the bridge at its highest point is 2*a*.	**Misconception:** - At the mid-point of *OR*, the value of *t* may not be the mid-value of 0 and π. Students who found $t = \frac{\pi}{2}$ at highest point of bridge by using mid-value of *t* = 0 and $t = \pi$ are penalized for this misconception.

<u>Alternative method 1:</u> Use $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ to solve for $t=\frac{\pi}{2}$ and find maximum $y=2a$. <u>Alternative method 2:</u> Use mid-value of *OR*, $x=\pi a$ and substitute into $x=a\left(2t-\sin 2t\right)$. But to solve $\pi=2t-\sin 2t$, need to “observe” that $t=\frac{\pi}{2}$.	Another misconception is that at maximum point, it is $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ and Not $\frac{\mathrm{d}y}{\mathrm{d}t}=0$ Note: a and α are two different symbols. Write correctly!
(ii) $\frac{\mathrm{d}x}{\mathrm{d}t}=a\left(2-2\cos 2t\right)$, $\frac{\mathrm{d}y}{\mathrm{d}t}=2a\sin 2t$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2a\sin 2t}{2a\left(1-\cos 2t\right)}=\frac{2\sin t\cos t}{1-\left(1-2\sin^2 t\right)}=\cot t$ (shown)	
(iii) Gradient $\frac{\mathrm{d}y}{\mathrm{d}x}=\cot t=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$ $\tan t=\sqrt{3}\ \Rightarrow t=\frac{\pi}{3}$ At *B*, $x=a\left(2\left(\frac{\pi}{3}\right)-\sin\frac{2\pi}{3}\right)=a\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$, $y=a\left(1-\cos\frac{2\pi}{3}\right)=a\left(1-\left(-\frac{1}{2}\right)\right)=\frac{3}{2}a$ $\therefore$ coordinates of *B* is $\left(a\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right),\ \frac{3}{2}a\right)$	Students need to note that $\frac{\pi}{6}$ is the gradient of line *AB* and Not the value of *t*. Students must evaluate the value of $\sin\frac{2\pi}{3}$ and $\cos\frac{2\pi}{3}$ and not to leave answer in trigonometry form. $\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$ and $\cos\frac{2\pi}{3}=-\frac{1}{2}$
(iv) Given *a* = 6, coordinates of *B* is $\left(6\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right),\ 9\right)$ $AP=\frac{9}{\tan 30^{\circ}}=9\sqrt{3}$  Exact area of the region bounded by the arch *ABCD* and the curve *BCR* 	**Students need to identify the correct region (in red). Use the correct formula, find area Not volume.** Need to find length of *AP* to find area of triangle *APB*. Coordinates of *B*, $6\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$ is only the length of *OP* not *AP*.

$= 2\left[\text{Area of triangle } ABP - \int_0^{\frac{\pi}{3}} 6(1-\cos 2t)(12-12\cos 2t)\,\mathrm{d}t \right]$	Students must remember that once the integral is change to with respect to ***t***, you need to change the upper and lower limit to be corresponding values of t and Not x.
$= 2\left[\frac{1}{2}(9)(9\sqrt{3}) - 72\int_0^{\frac{\pi}{3}} (1-\cos 2t)^2\,\mathrm{d}t \right]$	
$= 81\sqrt{3} - 144\int_0^{\frac{\pi}{3}} (1-\cos 2t)^2\,\mathrm{d}t \qquad \therefore h = 81\sqrt{3},\ k = 144, \alpha = \frac{\pi}{3}$	Students should state clearly what are the values of h, k and α .
$= 81\sqrt{3} - 144\int_0^{\frac{\pi}{3}} \left(1 - 2\cos 2t + \cos^2 2t\right)\mathrm{d}t$	
$= 81\sqrt{3} - 144\int_0^{\frac{\pi}{3}} \left(1 - 2\cos 2t + \frac{1+\cos 4t}{2}\right)\mathrm{d}t$	Need **to use double angle** before carrying out integration.
$= 81\sqrt{3} - 144\left[\frac{3}{2}t - \sin 2t + \frac{1}{8}\sin 4t\right]_0^{\frac{\pi}{3}}$	
$= 81\sqrt{3} - 144\left(\frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{1}{8}\left(-\frac{\sqrt{3}}{2}\right)\right)$	
$= 2\left(81\sqrt{3} - 36\pi\right)$ units^2	Answer must not be in terms of a.

11 In a research laboratory, scientists carry out experiments to produce a new substance C. C is produced when two substances A and B are reacted together in a chemical reaction. **At time t hours, the amount of C is x grams, and the amounts of A and B present are $(a-x)$ grams and $(b-x)$ grams respectively**, where a and b are real constants. At the start of the experiment, there are no traces of C found. At any instant, the **rate** at which the amount of C increases is **proportional** to the **product** of the amount of A and the amount of B present at that instant.

(i) In an experiment, a scientist carries out the experiment with $b = a$.

(a) Obtain a **differential equation relating x and t** and **solve for x in terms of t**. [4]

(b) Sketch the graph of x against t. [1]

(c) Given that there is $\frac{1}{2}a$ grams of C produced 1 hour after the experiment started, find the time needed for $\frac{4}{5}a$ grams of C to be produced. [2]

(ii) Another scientist carries out the experiment with $b < a$.

(a) Obtain a differential equation relating x and t and solve for x in terms of t. [4]

(b) Using your answer in part **(ii)(a)**, find the amount of C produced after a long time. [2]

[Solution]

(i) (a) $\frac{dx}{dt} = k(a-x)(b-x),\ k>0$

Given $b = a,\ \frac{dx}{dt} = k(a-x)^2,\ k>0$

$\int \frac{1}{(a-x)^2}\,dx = \int k\,dt$

$\frac{1}{a-x} = c + kt$

$x = a - \frac{1}{c+kt}$

When $t = 0$, $x = 0$, $c = \frac{1}{a}$

Be careful:

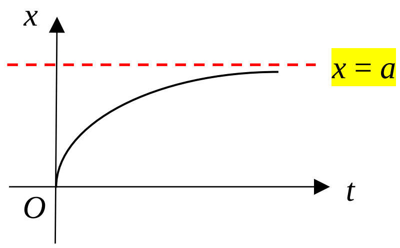
$\int (a-x)^{-2}\,dx$

$= \frac{(a-x)^{-1}}{(-1)(-1)} + c$

Given: At the start of the experiment, there are no traces of C found.

Thus $x = a - \frac{1}{\frac{1}{a}+kt}$ or $\frac{a^2kt}{1+akt}$ (Express x as the subject in terms of t)

(b)



Use a GC to sketch the **rectangular hyperbola** (subst say a = 2, k = 3). The hyperbola has a **horizontal asymptote x = a**

In this context, x and t are **non-negative**. Sketch curve only in **first quadrant**.

(c) When $t = 1$, $x = \frac{1}{2}a$, $\frac{1}{2}a = \frac{a^2k}{1+ak} \Rightarrow k = \frac{1}{a}$

Given: There is ½ a grams of C produced 1 hour after the experiment started

$\therefore\ x = \frac{at}{1+t}$

When $x = \frac{4}{5}a$, $\frac{4}{5}a = \frac{at}{1+t} \Rightarrow t = 4$

(ii) (a) Given $b < a$, $\frac{dx}{dt} = k(a-x)(b-x)$, $k > 0$

$$\int \frac{1}{(a-x)(b-x)}\,dx = \int k\ dt$$

Method 1 By Partial Fractions

$$\int \left(-\frac{1}{a-b}\right)\frac{1}{a-x} + \left(\frac{1}{a-b}\right)\frac{1}{b-x}\,dx = \int k\ dt$$

$$\left(-\frac{1}{a-b}\right)(-\ln(a-x)) + \left(\frac{1}{a-b}\right)(-\ln(b-x)) = kt + c$$

since $a-x>0$, $b-x>0$, $|a-x| = a-x$ and $|b-x| = b-x$

$$\frac{1}{a-b}\ln\left(\frac{a-x}{b-x}\right) = kt + c$$

$$\left(\frac{a-x}{b-x}\right) = e^{k(a-b)t+(a-b)c}$$

$$\frac{a-x}{b-x} = e^{k(a-b)t+c(a-b)} = Ae^{k(a-b)t}, \quad \text{where } A = e^{c(a-b)}$$

$$a - x = (b-x)Ae^{k(a-b)t}$$

$$x(Ae^{k(a-b)t} - 1) = bAe^{k(a-b)t} - a$$

$$x = \frac{bAe^{k(a-b)t} - a}{Ae^{k(a-b)t} - 1}$$

(Express x as the subject in terms of t)

Method 2 By Completing the Square

$$\int \frac{1}{(a-x)(b-x)}\,dx = \int k\ dt$$

$$\int \frac{1}{\left[x-\left(\frac{a+b}{2}\right)\right]^2-\left(\frac{a-b}{2}\right)^2}\,dx = \int k\ dt$$

$$\frac{1}{2\left(\frac{a-b}{2}\right)}\ln\left|\frac{x-\left(\frac{a+b}{2}\right)-\left(\frac{a-b}{2}\right)}{x-\left(\frac{a+b}{2}\right)+\left(\frac{a-b}{2}\right)}\right| = kt+c$$

$$\frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right| = kt+c$$

$$\frac{x-a}{x-b} = \pm e^{(a-b)kt+(a-b)c} = Ae^{(a-b)kt} \quad \text{where } A=\pm e^{(a-b)c}$$

$$x-a=(x-b)Ae^{(a-b)kt}$$

$$x=\frac{a-bAe^{(a-b)kt}}{1-Ae^{(a-b)kt}} \text{ or } \frac{bAe^{(a-b)kt}-a}{Ae^{(a-b)kt}-1}$$

(Express x as the subject in terms of t)

$$(a-x)(b-x)$$
$$=x^2-(a+b)x+ab$$
$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\left(\frac{a+b}{2}\right)^2+ab$$
$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{1}{4}\left(a^2+2ab+b^2\right)+ab$$
$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{1}{4}\left(a^2-2ab+b^2\right)$$
$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\left(\frac{a-b}{2}\right)^2$$

When $t=0$, $x=0$, $0=\frac{bAe^0-a}{Ae^0-1} \Rightarrow A=\frac{a}{b}$

Given: At the start of the experiment, there are no traces of C found.

Thus $x=\frac{a\left(e^{k(a-b)t}-1\right)}{\frac{a}{b}e^{k(a-b)t}-1}$

(b) $x=\frac{a\left(e^{k(a-b)t}-1\right)}{\frac{a}{b}e^{k(a-b)t}-1}\times\frac{e^{-k(a-b)t}}{e^{-k(a-b)t}}$

$$=\frac{a\left(1-e^{-k(a-b)t}\right)}{\frac{a}{b}-e^{-k(a-b)t}}$$

Using your answer in part (ii)(a)

As $t\to\infty$, $e^{k(a-b)t}\to\infty$ makes it difficult to find the limit of *x*.

So multiply both numerator and denominator by $e^{-k(a-b)t}$.

Then as $t\to\infty$, $e^{-k(a-b)t}\to 0$

Since $a-b>0$, as $t\to\infty$, $e^{-k(a-b)t}\to 0$, and so $x\to b$

Thus the amount of *C* produced after a long time is *b*.