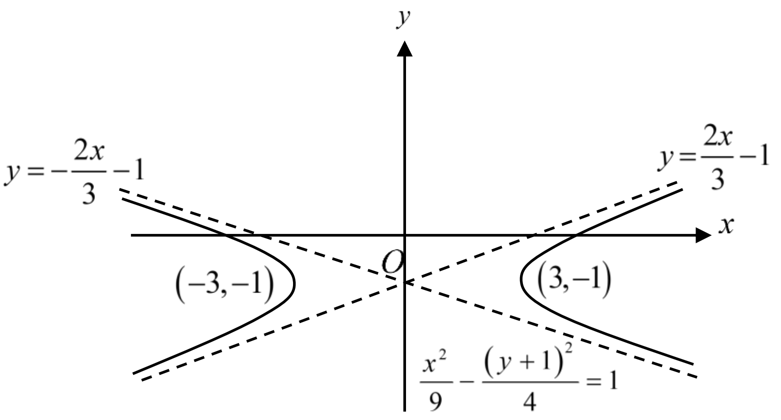
DHS 2022 Year 6 H2 Math Prelim Exam P2 solutions

Section A: Pure Mathematics [40 marks]

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td>1(a)</td><td>
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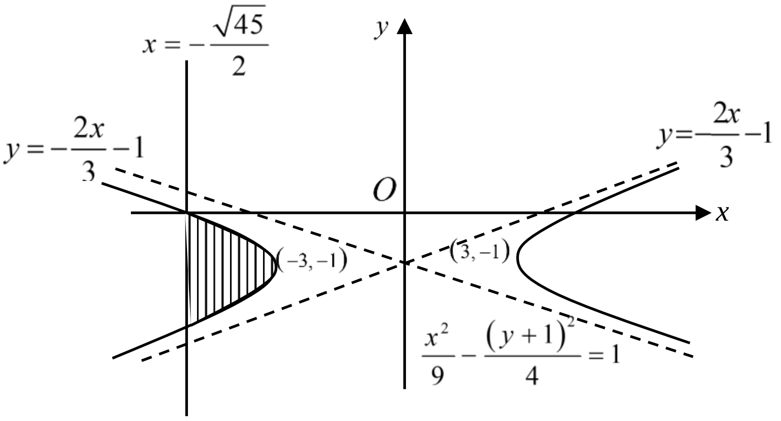
<tr><td>(b)</td><td>

When $x=-\frac{\sqrt{45}}{2}$
$\frac{(y+1)^2}{4}=\frac{x^2}{9}-1$
$\frac{(y+1)^2}{4}=\frac{9}{36}$
$y+1=\pm 1$
$y=-2$ or $y=0$

Required volume
$=\pi\left(\frac{\sqrt{45}}{2}\right)^2(2)-9\pi\int_{-2}^{0}\left(1+\frac{(y+1)^2}{4}\right)\mathrm{d}y$
$=9.42$ (to 3 s.f.)</td></tr>
</table>

	Alternative (for FM students only) Using shell method, $2\pi\int_{-3}^{-\frac{\sqrt{45}}{2}} x\left(2\sqrt{\frac{4x^2-36}{9}}\right)\mathrm{d}x = 9.42$ (to 3 s.f.)

Qn	Suggested Solution
2(a)	$w^3=(2-3\mathrm{i})^3$ $=(2)^3-3(2)^2(3\mathrm{i})+3(2)(3\mathrm{i})^2-(3\mathrm{i})^3$ $=8-36\mathrm{i}-54+27\mathrm{i}$ $=-46-9\mathrm{i}$
(b)	Since w is a root, $(2-3\mathrm{i})^3-5(2-3\mathrm{i})^2+a(2-3\mathrm{i})+b=0$ $(-46-9\mathrm{i})-5(4-12\mathrm{i}-9)+(2a+b-3a\mathrm{i})=0$ $(-21+2a+b)+(51-3a)\mathrm{i}=0$ $51-3a=0 \;\Rightarrow\; a=17$ $-21+2(17)+b=0 \;\Rightarrow\; b=-13$
(c)	Since w is a root of $z^3-5z^2+17z-13=0$, $w*$ is also a root as all the coefficients are real. $\therefore z^3-5z^2+17z-13=(z-w)(z-w^*)(z-\alpha)$ Replace z by $\frac{z}{\mathrm{i}}$, $\left(\frac{z}{\mathrm{i}}\right)^3-5\left(\frac{z}{\mathrm{i}}\right)^2+17\left(\frac{z}{\mathrm{i}}\right)-13=\left(\left(\frac{z}{\mathrm{i}}\right)-w\right)\left(\left(\frac{z}{\mathrm{i}}\right)-w*\right)\left(\left(\frac{z}{\mathrm{i}}\right)-\alpha\right)$ Multiply by i^3 on both sides, $z^3-5\mathrm{i}\,z^2-17z+13\mathrm{i}=(z-\mathrm{i}w)(z-\mathrm{i}w^*)(z-\mathrm{i}\alpha)$ A possible cubic polynomial is $z^3-5\mathrm{i}\,z^2-17z+13\mathrm{i}$.

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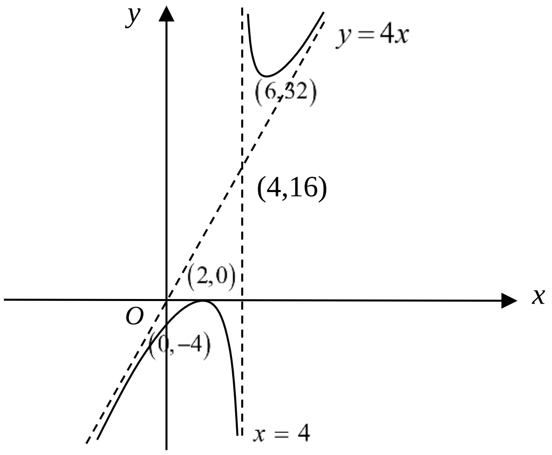
<tr><td>3(a)</td><td>
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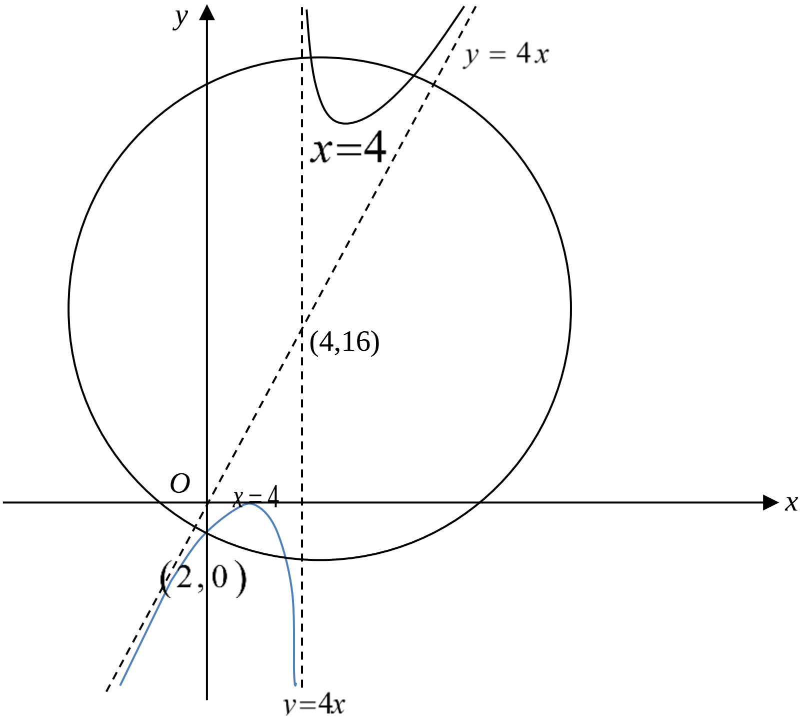
<tr><td>(b)</td><td>

$$(x-4)^2+\left(\frac{4(x-2)^2}{x-4}-16\right)^2=a$$

$$(x-2)^2+(y-16)^2=\left(\sqrt{a}\right)^2$$

Therefore, the solution to the equation is the intersection between $y=\frac{4(x-2)^2}{x-4}$ and circle with radius $\sqrt{a}$ centred at $(4,16)$ which is the point of intersection between the 2 asymptotes.

Distance between $(4,16)$ and $(0,-4)$</td></tr>
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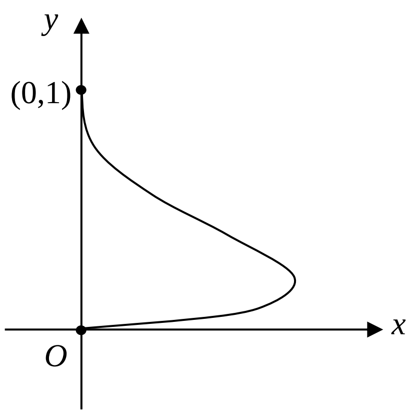
	$=\sqrt{(4-0)^2+(16+4)^2}=\sqrt{416}$ Hence, $a>416$ so that the equation will have 1 negative real root.
(c)	Intersection point of the 2 asymptotes is (4, 16) Thus, $\tan^{-1}(4)<\arg(z-4-16\mathrm{i})<\frac{\pi}{2}$ Or $-\left(\pi-\tan^{-1}(4)\right)<\arg(z-4-16\mathrm{i})<-\frac{\pi}{2}$

Qn	Suggested Solutions
4(a)	$\overrightarrow{OA}=\frac{\mu\overrightarrow{OQ}+(1-\mu)\overrightarrow{OP}}{\mu+(1-\mu)}$ $=\mu\begin{pmatrix}0\\2\\-t\end{pmatrix}+(1-\mu)\begin{pmatrix}1\\0\\0\end{pmatrix}$ $=\begin{pmatrix}1-\mu\\2\mu\\-t\mu\end{pmatrix}$ $\overrightarrow{OB}=\frac{\mu\overrightarrow{OR}+(1-\mu)\overrightarrow{OQ}}{\mu+(1-\mu)}$ $=\mu\begin{pmatrix}0\\0\\t\end{pmatrix}+(1-\mu)\begin{pmatrix}0\\2\\-t\end{pmatrix}$ $=\begin{pmatrix}0\\2-2\mu\\-t+2t\mu\end{pmatrix}$ $\overrightarrow{AB}=\begin{pmatrix}0\\2-2\mu\\-t+2t\mu\end{pmatrix}-\begin{pmatrix}1-\mu\\2\mu\\-t\mu\end{pmatrix}=\begin{pmatrix}\mu-1\\2-4\mu\\-t+3t\mu\end{pmatrix}$
(b)	Clearly, $\overrightarrow{OA}\neq k\overrightarrow{OB}$ This means the points are not collinear.

(c)	$$\frac{\left\lvert \overrightarrow{AB} \bullet \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\rvert} = \frac{1}{5}$$ $$\frac{\left\lvert \begin{pmatrix} \mu-1 \\ 2-4\mu \\ -t+3t\mu \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\rvert} = \frac{1}{5}$$ $3\mu-3+8-16\mu=\pm 1$ $13\mu = 4$ or 6 $\mu = \frac{4}{13}$ or $\frac{6}{13}$
(d)	If angle AOB is a right angle, then $\overrightarrow{OA} \bullet \overrightarrow{OB} = 0$ $$\begin{pmatrix} 1-\mu \\ 2\mu \\ -t\mu \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2-2\mu \\ -t+2t\mu \end{pmatrix} = 0$$ $4\mu-4\mu^2+t^2\mu-2t^2\mu^2=0$ **<u>Method 1</u>** $4\mu-4\mu^2+t^2\mu-2t^2\mu^2=0$ $\mu=0$ or $4-4\mu+t^2-2t^2\mu=0$ (reject $\because 0<\mu<1$) $\mu = \frac{4+t^2}{4+2t^2}$ Clearly, $\frac{4+t^2}{4+2t^2} > 0$ since $4+t^2>0$ and $4+2t^2$ for all $t \in \mathbb{R}$. Since $0<\mu<1$, $\frac{4+t^2}{4+2t^2} < 1$ $4+t^2 < 4+2t^2$ $t^2 > 0$ Hence $t \in \mathbb{R} \setminus \{0\}$

	Method 2 Since $0<\mu<1$, $4-4\mu+t^2-2t^2\mu=0$ $t^2=\dfrac{4(1-\mu)}{2\mu-1}$ From the graph of t^2 vs μ for $0<\mu<1$, $t^2>0$ or $t^2<-4$ (no solutions for t) then $t\in\mathbb{R}\setminus\{0\}$.

Qn	Suggested Solution
5(a)	$\int \sin x(1-\sin x)\ \mathrm{d}x$ $=\int \sin x-\sin^2 x\ \mathrm{d}x$ $=\int \sin x-\dfrac{1}{2}(1-\cos 2x)\ \mathrm{d}x$ $=-\cos x-\dfrac{1}{2}x+\dfrac{1}{4}\sin 2x+D$ (shown)
(b)	 y-intercepts: $x=\theta(1-\sin\theta)=0$ $\theta=0$ or $\sin\theta=1\Rightarrow\theta=\dfrac{\pi}{2}$ $\left(\because 0\le t\le\frac{\pi}{2}\right)$ When $\theta=0$, $y=1-\cos 0=0$ When $\theta=\dfrac{\pi}{2}$, $y=1-\cos\dfrac{\pi}{2}=1$

(c)

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=1-\sin\theta-\theta\cos\theta,\quad \frac{\mathrm{d}y}{\mathrm{d}\theta}=\sin\theta$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\sin\theta}{1-\sin\theta-\theta\cos\theta}$$

Since tangent at P is parallel to $y=2x$

$$\frac{\sin\theta}{1-\sin\theta-\theta\cos\theta}=2$$

$$3\sin\theta+2\theta\cos\theta-2=0$$

From GC : $\theta=0.42230\quad\left(\because 0\le t\le\frac{\pi}{2}\right)$

$\therefore P\left(0.249,\ 0.0879\right)$

(d)

Area enclosed

$$=\int_0^1 x\ \mathrm{d}y$$

$$=\int_0^{\frac{\pi}{2}}\theta(1-\sin\theta)\cdot(\sin\theta\ \mathrm{d}\theta)$$

$$=\int_0^{\frac{\pi}{2}}\theta\ \left[\sin\theta(1-\sin\theta)\right]\ \mathrm{d}\theta$$

$$\boxed{\begin{array}{ll} u=\theta, & \dfrac{\mathrm{d}v}{\mathrm{d}\theta}=\sin\theta(1-\sin\theta) \\ \dfrac{\mathrm{d}u}{\mathrm{d}\theta}=1, & v=-\cos\theta-\dfrac{\theta}{2}+\dfrac{\sin 2\theta}{4} \end{array}}$$

$$=\left[\theta\left(-\cos\theta-\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\left(-\cos\theta-\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\ \mathrm{d}\theta$$

$$=\left[\theta\left(-\cos\theta-\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\right]_0^{\frac{\pi}{2}}+\left[\sin\theta+\frac{\theta^2}{4}+\frac{\cos 2\theta}{8}\right]_0^{\frac{\pi}{2}}$$

$$=\left[\frac{\pi}{2}\left(-\frac{\pi}{4}\right)\right]+\left[\left(1+\frac{\pi^2}{16}-\frac{1}{8}\right)-\frac{1}{8}\right]$$

$$=\frac{3}{4}-\frac{\pi^2}{16}$$

(e)

Gradient of $L=2$

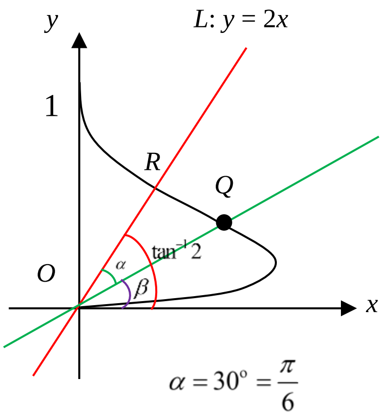
$\Rightarrow$ angle btw L and $x-$axis $=\tan^{-1}2$

$$\beta=\tan^{-1}2-\alpha$$

$$\tan^{-1}\frac{y}{x}=\tan^{-1}2-\frac{\pi}{6}$$

$$\tan^{-1}\frac{1-\cos\theta}{\theta(1-\sin\theta)}=\tan^{-1}2-\frac{\pi}{6}$$

From GC : $\theta=0.596$ rad. (3 sf)

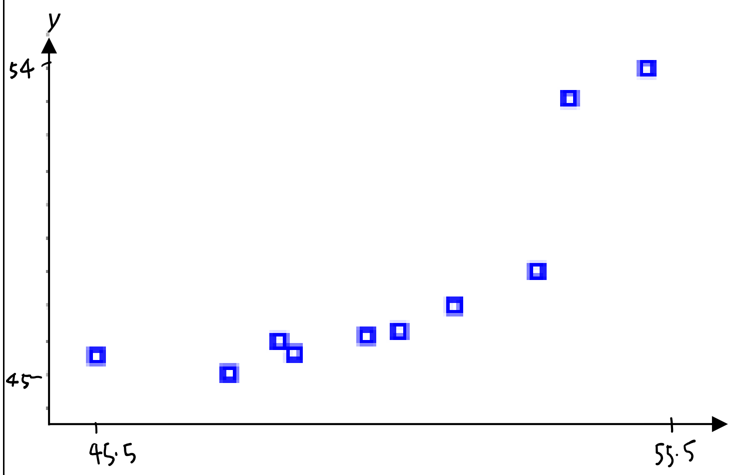


	Alternative Use dot product, $\cos\frac{\pi}{6} = \frac{\overrightarrow{OR}\bullet\overrightarrow{OQ}}{\left|\overrightarrow{OR}\right|\left|\overrightarrow{OQ}\right|} = \frac{\begin{pmatrix}1\\2\\0\end{pmatrix}\begin{pmatrix}\theta(1-\sin\theta)\\1-\cos\theta\\0\end{pmatrix}}{\left|\begin{pmatrix}1\\2\\0\end{pmatrix}\right|\left|\begin{pmatrix}\theta(1-\sin\theta)\\1-\cos\theta\\0\end{pmatrix}\right|} = \frac{\theta(1-\sin\theta)+2(1-\cos\theta)}{\sqrt{5}\ \sqrt{\left(\theta(1-\sin\theta)\right)^2+\left(1-\cos\theta\right)^2}}$ From GC : $\theta = 0.596$ rad. (3 sf) **Note**: The other case where $\tan^{-1}2+\alpha > \frac{\pi}{2}$ need not be considered as there would be no solution.

Section B: Probability and Statistics [60 marks]

Qn	Suggested Solution
6(a)	Ways = $(7-1)!\times {}^7C_3\times 3!$ = 151200
b(i)	T Y R A N O S U R A N **<u>Case 1</u>** All 5 different letters (ie. No identical) = ${}^8C_5\times 5!$ = 6720 **<u>Case 2</u>** 2 identical (RR, AA or NN) = ${}^3C_1\times {}^7C_3\times \dfrac{5!}{2!}$ = 6300 Total ways = 6720 + 6300 = 13020
b(ii)	**<u>Method 1</u>** Reduced sample space = $\dfrac{1}{{}^8C_2}=\dfrac{1}{28}$ **<u>Method 2a</u>** Conditional probability $=\dfrac{\text{P(2R2N} \cap \text{"RAN")}}{\text{P("RAN")}}$ $=\dfrac{\text{no. of ways (2R2N} \cap \text{"RAN")}}{\text{no. of ways ("RAN")}}$ = $=\dfrac{3!}{{}^8C_2\times 3!}$ $=\dfrac{1}{28}$ **<u>Method 2b</u>** Conditional probability $=\dfrac{\text{P(2R2N} \cap \text{"RAN")}}{\text{P("RAN")}}$ $=\dfrac{\frac{2}{11}\times\frac{2}{10}\times\frac{2}{9}\times\frac{1}{8}\times\frac{1}{7}\times 3!}{\frac{2}{11}\times\frac{2}{10}\times\frac{2}{9}\times\frac{1}{8}\times\frac{1}{7}\times {}^8C_2\times 3!}$ $=\dfrac{1}{28}$

Qn	Suggested Solution
7(a)(i)	$\mathrm{P}(A\cap B)$ $=\mathrm{P}(\text{fall, rise, rise})+\mathrm{P}(\text{fall, fall, rise})$ $=(0.4\times0.15\times0.6)+(0.4\times0.85\times0.15)$ $=0.087$
(ii)	$\mathrm{P}(B)$ $=\mathrm{P}(A\cap B)+\mathrm{P}(A'\cap B)$ $=0.087+\mathrm{P}(\text{rise, rise, rise})+\mathrm{P}(\text{rise, fall, rise})$ $=0.087+(0.6\times0.6\times0.6)+(0.6\times0.4\times0.15)$ $=0.339$
(iii)	$\mathrm{P}(B\mid A)$ $=\dfrac{\mathrm{P}(B\cap A)}{\mathrm{P}(A)}$ $=\dfrac{0.087}{0.4}$ $=0.2175$
(b)	Since $\mathrm{P}(B\mid A)=0.2175\neq0.339=\mathrm{P}(B)$, A and B are not independent.
(c)	Let W be the number of Tuesdays in which the unit price of X rises, out of 12 Tuesdays. $W\sim \mathrm{B}(12,0.6)$ $\mathrm{P}(W=5)=0.101\ (3\text{ s.f.})$

Qn	Suggested Solution
8(a)	• Set B will have a larger $\lvert r \rvert$. • The data points for **Set *B* lie relatively closer to a straight line** with negative gradient whereas **Set *A*'s $\lvert r \rvert$ value will be closer to 0** since the data points are more scattered with weak linear correlation between x and y.
(b)(i)	
(ii)	Model C: r = 0.81730 = 0.817 (3 sf) Model D: r = 0.93944 = 0.939 (3 sf) Since $\lvert r \rvert$ value for model D is closer to 1 compared to model C, it indicates a stronger linear correlation. Hence model D is more appropriate.
(iii)	Equation of regression line of y on x for Model D: $y = 45.423 + 8.5357 \times 10^{-12} e^{\frac{1}{2}x}$ When $x = 50$, $y = 46.0376$ The mean household expenditure is estimated to be **\$46 038** The estimate is reliable since it is an **interpolation** where x = 50 ($45.5 \le x \le 55.5$) and **$\lvert r \rvert$ is close to 1** which indicates a strong positive linear correlation between x and y.
(iv)	It is not valid because correlation between income and expenditure does not imply causation.
(v)	Not true, as the product moment correlation coefficient for $y = a + be^{\frac{1}{2}x}$ measures the linear correlation between y and $e^{\frac{1}{2}x}$, not y and $e^{\frac{1}{20}x}$. **<u>Alternative</u>** Not true. For $y = p + qe^{\frac{1}{20}x}$, the product moment correlation coefficient is 0.8639998 = 0.864 (3.s.f), which is different.

Qn	Suggested Solution
9(a)	Let X be the mass of a randomly chosen mooncake. $H_0 : \mu = 150$ $H_1 : \mu < 150$ where μ is the population mean mass of mooncakes. Since sample size of 9 is small, assume X follows a normal distribution. Under H_0, $\bar{X} \sim N\left(150, \frac{6.73^2}{9}\right)$ From GC, p-value = 0.186322 = 0.186 (3 s.f.) Since the p-value > 0.1, we <u>do not reject H_0 and conclude that t</u>here is insufficient evidence at the 10% significance level that the mean mass of the mooncake is less than 150 g, i.e. insufficient evidence to reject owner's claim.
9(b)	Let Y be the working hours of a randomly chosen teacher in the school. $s^2 = \frac{n}{n-1}(\text{sample variance}) = \frac{50k^2}{49}$ hours^2 $H_0 : \mu = 60$ $H_1 : \mu \neq 60$ Under H_0, $\bar{Y} \sim N\left(60, \frac{k^2}{49}\right)$ approximately by Central Limit Theorem since sample size of 50 is large. In order to reject H_0, p-value = $2\,P(\bar{Y} \geq 62) \leq 0.05$ From GC (graph), $0 < k \leq 7.14299$ Set of values of k is $\{k \in \mathbb{R} : 0 < k \leq 7.14\}$. **<u>Alternative</u>** In order to reject H_0, $\bar{y}$ must lie within the critical region. i.e, $\bar{y} \geq \bar{y}_{\text{critical}}$ $\therefore \bar{y}_{\text{critical}} \leq 62$ From GC (graph), $0 < k \leq 7.14$ (to 3sf) Set of values of k is $\{k \in \mathbb{R} : 0 < k \leq 7.14\}$.

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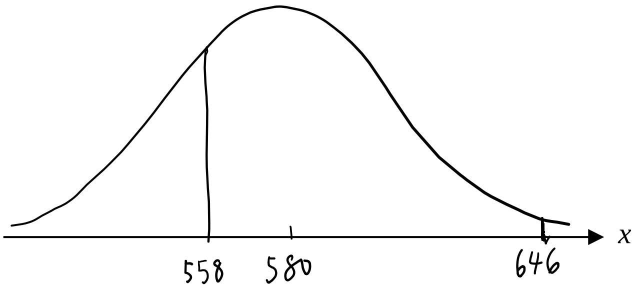
10(a)	
(b)	$X \sim \text{N}(580, 22^2)$ Expected number $= 300 \times \text{P}(X > 600)$ $= 300 \times 0.18165$ $= 54.495$ $= 54.5$ (3 s.f.)
(c)	No. By combining the masses, it would give a distribution with 2 peaks instead of a single peak.
(d)	Let K and L be the selling price of a randomly chosen rock melon and watermelon respectively. $K = 0.003X, \quad L = 0.0028Y$ $K \sim \text{N}(0.003 \times 580,\ 0.003^2 \times 22^2)$ $K \sim \text{N}(1.74, 0.004356) \Rightarrow \bar{K} \sim \text{N}\left(1.74, \dfrac{0.004356}{4}\right)$ $L \sim \text{N}(0.0028 \times 870,\ 0.0028^2 \times 30^2)$ $L \sim \text{N}(2.436, 0.007056)$ $\bar{K} - L \sim \text{N}(-0.696, 0.008145)$ $\text{P}(\lvert \bar{K} - L \rvert \le 0.60)$ $= \text{P}(-0.60 \le \bar{K} - L \le 0.60)$ $= 0.14373$ $= 0.144$ (3s.f.)
(e)	$K_1 + ... + K_n \sim \text{N}(1.74n, 0.004356\text{n})$ $L_1 + ... + L_{20-n} \sim \text{N}(2.436(20-n), 0.007056(20-n))$ Let W be the total cost of the 20 melons. $W = K_1 + ... + K_n + L_1 + ... + L_{20-n}$ $W \sim \text{N}(1.74n + 2.436(20-n), 0.004356n + 0.007056(20-n))$ $\text{P}(W > 38) > 0.95$ Using GC table, $n = 13, \quad \text{P}(W > 38) = 1 > 0.95$ $n = 14, \quad \text{P}(W > 38) = 0.9988 > 0.95$ $n = 15, \quad \text{P}(W > 38) = 0.8113 < 0.95$ Greatest n = 14

Qn	Suggested Solution
11(a)	$\sum_{r=1}^{\infty} \mathrm{P}(X=r)=1$ $\sum_{r=1}^{\infty} \frac{a}{r^3}=1$ $a=\frac{1}{1.2021}=0.83188=0.832$ (3 sf)
(b)	$\mathrm{E}(X)=\sum_{r=1}^{\infty} r\ \mathrm{P}(X=r)=a\sum_{r=1}^{\infty}\frac{1}{r^2}=1.37$ (3 s.f) $\mathrm{E}(X^2)=\sum_{r=1}^{\infty} r^2\ \mathrm{P}(X=r)=a\sum_{r=1}^{\infty}\frac{1}{r}$ does not exist. Therefore Var(X) cannot be calculated.
(c)	Method 1 $\mathrm{P}\left(X\geq 2 \mid X\leq 15\right)$ $=1-\mathrm{P}\left(X=1 \mid X\leq 15\right)$ $=1-\frac{\mathrm{P}\left(X=1\right)}{\mathrm{P}\left(X\leq 15\right)}$ $=1-\frac{a}{\sum_{r=1}^{15}\frac{a}{r^3}}$ $=0.16665=0.167$ (3 s.f) Method 2 $\mathrm{P}\left(X\geq 2 \mid X\leq 15\right)$ $=\frac{\mathrm{P}\left(2\leq X\leq 15\right)}{\mathrm{P}\left(X\leq 15\right)}$ $=\frac{\sum_{r=2}^{15}\frac{a}{r^3}}{\sum_{r=1}^{15}\frac{a}{r^3}}$ $=0.16665=0.167$ (3 s.f)
(d)	$Y\sim \mathrm{B}(10,\mathrm{P}(X=3))$ where $\mathrm{P}(X=3)=\frac{a}{27}=\frac{0.83188}{27}=0.030810$ $\mathrm{P}(Y>2)=1-\mathrm{P}(Y\leq 2)=0.00298$
(e)	Note: X_1 and Y are dependent variables. **<u>Case 1:</u>** $X_1=1$ and $Y=2$ The first number must be a '1' and the rest of 9 numbers must have two '3's. $\mathrm{P}(X=1)\left[\binom{9}{2}\left(\mathrm{P}(X=3)\right)^2\left(1-\mathrm{P}(X=3)\right)^7\right]=0.022835$ **<u>Case 2:</u>** $X_1=2$ and $Y=1$ The first number must be a '2' and the rest of 9 numbers must have one '3'. $\mathrm{P}(X=2)\left[\binom{9}{1}\left(\mathrm{P}(X=3)\right)\left(1-\mathrm{P}(X=3)\right)^8\right]=0.022448$ **<u>Note:</u>** $X_1=3$ and $Y=0$ This case is impossible as Y is counting the number of '3' generated, probability is 0 for this case. $\therefore \mathrm{P}(X_1+Y=3)=0.022835+0.022448+0=0.0453$ (3 sf)