RAFFLES INSTITUTION
2022 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 22

MATHEMATICS **9758/01**

Paper 1 **3 hours**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper. **You may use the blank page on page 2 if necessary and you are reminded to indicate the question number(s) clearly.**
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

For Examiner's Use Only						
Q1	Q2	Q3	Q4	Q5	Q6	Q7
/ 6	/ 6	/ 7	/ 7	/ 8	/ 9	/ 9
Q8	Q9	Q10	Q11		TOTAL	
/ 10	/ 11	/ 13	/ 14		/ 100	

This document consists of **23** printed pages and **1** blank page.

BLANK PAGE

You may continue your working on this page if necessary, indicating the question number(s) clearly.

1 **(i)** On the same axes, sketch the graphs of $y = \dfrac{x}{x-a}$ and $y = 2|x-a|+1$, where a is a positive constant. [3]

(ii) Hence solve the inequality $\dfrac{x}{x-a} > 2|x-a|+1$, leaving your answer in terms of a. [3]

2 A function is defined as $\mathrm{f}(x)=\dfrac{1+2x-x^2}{2(x^2-2x)}, \quad x\in\mathbb{R}, x\neq 0, 2.$

(i) Show that $\mathrm{f}(x)$ can be written in the form $q\left(\dfrac{2-(x+p)^2}{(x+p)^2-1}\right)$, where p and q are constants to be found. [2]

(ii) Hence describe a sequence of transformations that will transform the graph of $y=\dfrac{2-x^2}{x^2-1}$ onto the graph of $y=\mathrm{f}(x)$. [2]

(iii) Determine, with the help of a sketch or otherwise, the set of values of k for which the equation $\dfrac{2-x^2}{x^2-1}=k$ has no real roots. [2]

3 A curve has parametric equations

$$x = a\left(1 + \frac{1}{t}\right), \quad y = a\left(t - \frac{1}{t^2}\right),$$

where a is a constant and $t \neq 0$.

(i) Find the equations of the tangent and the normal to the curve at the point P where $t = -\frac{1}{2}$. [5]

(ii) The tangent at P meets the y-axis at Q and the normal at P meets the y-axis at R. Show that the area of triangle PQR is $\dfrac{241}{120}a^2$. [2]

4 **(a)** The point R has position vector $\mathbf{r}$. Given that $\mathbf{r} = \begin{pmatrix} a \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$, where a is a real number, describe geometrically the set of all possible positions of the point R, as a varies. [2]

(b) **(i)** The points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively. Show that the point F, the foot of perpendicular from the origin O to the line passing through P and Q, has position vector $(1-\lambda)\mathbf{p} + \lambda\mathbf{q}$, where

$$\lambda = \frac{|\mathbf{p}|^2 - \mathbf{p}.\mathbf{q}}{|\mathbf{q} - \mathbf{p}|^2}.$$ [4]

(ii) Write down an inequality satisfied by λ for F to lie within the line segment PQ. [1]

5 **Do not use a calculator in answering this question.**

The complex numbers z and w are given by

$$z = \sin\left(\frac{\pi}{6}\right) + \mathrm{i}\cos\left(\frac{\pi}{6}\right) \text{ and } w = \sqrt{2}\left[\sin\left(\frac{\pi}{6}\right) + \mathrm{i}\cos\left(\frac{\pi}{3}\right)\right].$$

(i) Find $|z|$ and $\arg(z)$. Hence find the value of z^3. [3]

(ii) By considering some suitable form of w or otherwise, find w^4. [3]

(iii) Hence find the value of $z^{2022} - w^{2020}$. [2]

6 **(a)** A sequence is such that $u_1 = p$, where p is a constant, and $u_{n+1} = \frac{4}{u_n}$, for $n > 0$.

Describe how the sequence behaves when

(i) $p = 2$, [1]

(ii) $p = 3$. [1]

(b) Another sequence $v_1,\ v_2,\ v_3, \ldots$ is such that $v_n = v_{n-1} + n$, where $n \geq 2$, and $v_1 = A$.

Find, in terms of A and n, an expression for

(i) v_n, [4]

(ii) $\sum_{r=1}^{n} v_r$. [3]

(You need not simplify your answer. You may use the result $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.)

7 It is given that $y=\sqrt{2+\cos^2 x}$.

(a) Show that

(i) $2y\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\sin 2x$, [1]

(ii) $y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-\cos 2x$. [1]

(b) Hence find the Maclaurin series of y, up to and including the term in x^4. [5]

(c) By substituting $x=\frac{\pi}{6}$, show that $\sqrt{11}\approx 2\sqrt{3}\left(1-\frac{\pi^2}{216}+\frac{\pi^4}{31104}\right)$. [2]

8 A curve C is defined by $y = \dfrac{(\ln x)^4}{\sqrt{x}}$ where $0 < x < 10$.

(i) Find the exact volume generated when the area bounded by C, the x-axis and the lines $x = 1$ and $x = \mathrm{e}$ is rotated about the x-axis through 360°. [4]

(ii) Find the area enclosed by C and the lines $y = 1$ and $y = \mathrm{e}$. [6]

9 The functions f, g and h are defined as follows:

$$\mathrm{f}: x \mapsto x^3 - 1, \quad x \in \mathbb{R},$$
$$\mathrm{g}: x \mapsto \mathrm{e}^{-3x}, \quad x \in \mathbb{R}, x < 0,$$
$$\mathrm{h}: x \mapsto \frac{x+1}{x-1}, \quad x \in \mathbb{R}, x \neq 1.$$

(i) Define in a similar form, the inverse functions g^{-1} and h^{-1}. [4]

(ii) Write down the rule of the composite function ff. [1]

(iii) It is known that the equation $\mathrm{ff}(x) = 0$ has only one real root α.

Find the exact value of α and hence show that $\mathrm{g}^{-1}(\alpha) = k \ln 2$, where k is a constant to be found. [2]

(iv) Show that the composite function hg exists and write down the range of this function. [2]

(v) Denoting composite functions hh as h^2, hhh as h^3 and so on, find the value of x for which $\mathrm{h}^m(x) = \mathrm{h}^{-1}(-1)$, where m is a positive even integer. [2]

10 Glucose in the blood stream is reduced at a rate proportional to the amount of glucose present in the blood stream.

Let G milligrams (mg) be the amount of glucose in 1 decilitre (dL) of blood stream at time t minutes and let λ denote the positive constant of proportionality.

(i) Write down a differential equation relating G, λ and t. Solve this differential equation to find an expression of G in terms of λ and t. [3]

Through extensive research, doctors recommended that a healthy glucose level in the blood stream of an adult should be between 70 mg/dL to 100 mg/dL.

An adult patient, Neo, has 80 mg/dL of glucose in his blood stream at time $t = 0$ minutes. Due to a medical condition, he is not able to extract glucose from the food he eats. As he is not able to replenish the glucose in his blood stream, he is thus losing his glucose in the blood stream at a rate (in mg/dL per minute) corresponding to $\lambda = 0.005$ per minute.

(ii) Find the approximate time it would take for Neo's glucose level in the blood stream to fall below the healthy range if there is no intake of glucose. [2]

In order to maintain Neo's glucose level in the blood stream in the healthy range, glucose is injected into his blood stream intravenously at a constant rate of μ mg/dL per minute.

(iii) Write down a differential equation relating G, μ and t to model Neo's situation. Solve this differential equation to find an expression of G in terms of μ and t. [5]

(iv) Explain whether it is recommended to keep injecting glucose at a rate of 0.7 mg/dL per minute into Neo's blood stream. [2]

(v) Find the maximum range of values of μ so that Neo's glucose level in the blood stream will always be in the healthy range. [1]

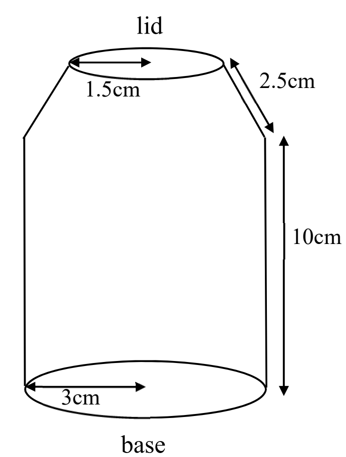
11 [It is given that 1 ml = 1 cm^3 and that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]

Globally, 2 trillion drink cans are produced every year. Drink cans constitute part of the packaging cost for beverage companies and using the appropriate material in appropriate amounts will enable them to save costs. A new beverage company wants to produce drink cans using a particular material for the top and bottom of a cylindrical shaped drink can, and another material for the curved body. For a fixed thickness, the material for the top and bottom costs $1.20/$\text{m}^2$ and the material for the curved side costs $0.90/$\text{m}^2$.

(i) For a fixed volume of 100π ml per can, show, using differentiation, that the radius r of the most economical can is approximately 3.35 cm and evaluate the corresponding height h. [7]

(ii) Hence, find the cost of the most economical can, giving your answer correct to the nearest 0.1 cent. [1]

In order to make the can more attractive, the company redesigns the can such that the base radius of the can is 3 cm and the height of the cylindrical part is 10 cm. At the top of each can, there is a tapering in before being covered by a circular lid of radius 1.5 cm, with the dimensions of the can as shown below.



(iii) The beverage is pumped into the can at a rate of 90π ml/s. Find the rate at which the liquid level in the can is increasing when it is 1 cm from the lid of the can. [6]

11 [Continued]