HWA CHONG INSTITUTION

2020 JC2 Preliminary Examination

Higher 2

9758/01

MATHEMATICS

28 August 2020

3 hours

Candidate Name			CT Group	

Centre Number	S					Index Number				

Write here how many additional pieces of writing paper you have used (if any).	

Candidates answer on the Question Paper.

Additional Materials: List of Formulae 26 (MF26)

READ THESE INSTRUCTIONS FIRST

1. Write your name and class on this Cover Page and any additional writing paper you hand in.
2. Write in dark blue or black pen.
3. You may use an HB pencil for any diagrams or graphs.
4. **Do not use staples,** paper clips, highlighters, glue or **correction fluid.**
5. Answer **all** the questions and write your answers in the spaces provided in the Question Paper.
6. Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
7. Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
8. Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
9. You are reminded of the need for clear presentation in your answers.
10. The number of marks is given in brackets [ ] at the end of each question or part question.

For Examiner's Use			
Qn	**Marks**	**Total**	**Remarks**
1		4	
2		5	
3		6	
4		7	
5		7	
6		8	
7		8	
8		9	
9		10	
10		10	
11		12	
12		14	
		100	

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1 A curve has parametric equations

$$x = 2t - \frac{1}{t} \quad \text{and} \quad y = t + \frac{1}{t}$$

where t is a parameter and $t \neq 0$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{1}{2} - \dfrac{3}{2\left(2t^2 + 1\right)}$. [2]

(ii) Find the range of values that $\dfrac{dy}{dx}$ can take. [2]

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2 The coefficient of x^3 in the expansion of $\frac{1}{(2+ax)^5}$ is 5 times the coefficient of x^2 in the expansion of $\sqrt{4-ax}$ where a is a non-zero constant. Find the exact value of a. [5]

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3 A curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{2(x-1)^2}{x-3}$.

(i) Sketch C, showing clearly the equations of the asymptotes and coordinates of any stationary points and axial intercepts. [3]

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(ii) Sketch the graph of $y = \mathrm{f}\left(|x|\right)$ on a separate diagram. It is given that the equation $x^2 + \left[\dfrac{2\left(|x|-1\right)^2}{|x|-3} - k\right]^2 = a^2$, where $a > 0$ and $k > 0$, has an odd number of roots. By adding a suitable curve to the graph of $y = \mathrm{f}\left(|x|\right)$, find a in terms of k. [3]

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4 **(a)** Find the exact solution of the inequality $\dfrac{2x+1}{x^2-1} \le 1$. [3]

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(b) A curve G has equation $ay^2 + bx^3 + cx = 2,$ where a, b and c are constants. It is given that G passes through the point $\left(1, \sqrt{3}\right)$ and the gradient at the point $\left(-1, 1\right)$ is $-\dfrac{3}{2}.$ Find the equation of the curve G. [4]

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5 A closed right circular cone of base radius r and height h has a fixed total surface area $k\pi$ units^2, where k is a positive constant. Show that $r^2 = \frac{k^2}{\left(h^2+2k\right)}$. Given that r and h vary, find the greatest possible volume of the cone, giving your answer in exact form in terms of k. [7]

[The volume of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$ and the curved surface area of the cone is given by $A = \pi rl$, where l is the slant height .]

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6 Find

(a) $\displaystyle\int \frac{3e^x}{5-0.3e^x}\,dx$, [2]

(b) $\displaystyle\int \cos(\ln x)\,dx$, where $x>0$, [3]

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(c) the exact value of $\int_0^3 \left|2e^x - 5\right| \, dx$. [3]

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7 A curve P is defined by $y = -\sin^2 x$ where $-\pi \le x \le 0$.

(i) Find the exact coordinates of the minimum point of P. [2]

[There is no need to show that the point is a minimum point.]

The finite region R is bounded by the curve P, the y-axis and the lines $x = -\dfrac{\pi}{2}$, $y = -2$.

Find

(ii) the exact area of R, [3]

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(iii) the volume generated when R is rotated through 2π radians about the $y-$axis. [3]

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8 **Do not use a graphing calculator in answering this question.**

The complex number w is a root to the equation

$$z^4 + az^3 + 46z^2 + bz + 125 = 0,$$

where $a, b \in \mathbb{R}$.

(i) Prove that the product of the factors $z - w$ and $z - w^*$ is a quadratic polynomial with real coefficients. [2]

(ii) Given that $|w| = \sqrt{5}$ and $\arg(w) = -\tan^{-1}(2)$ where $-\frac{\pi}{2} < -\tan^{-1}(2) < 0$, show that $w = 1 - 2\text{i}$. [2]

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(iii) Find all the roots of the equation and the values a and b. [5]

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9 Oil spill from a boat has a significant impact on the population of fish in a lake. The population (in thousands) t days after an oil spill is denoted by x. Statistics show that the population increases at a rate inversely proportional to the square of the population and that fish die, due to the oil spill, at a rate proportional to the population. When $x = 0.5$, the number of fish remains constant.

(i) Show that $\frac{dx}{dt} = \frac{k(1-8x^3)}{x^2}$, where k is a positive constant. [3]

(ii) When the oil spill occurred, there were 3000 fish in the lake. One day after the oil spill, the population became 2000. Show that $1-8x^3 = -215\left(\frac{63}{215}\right)^t$. Find the number of fish on the third day after the oil spill, giving your answer to the nearest integer. [5]

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(iii) Explain, with justification, what will happen to the population of fish in the lake in the long run. [2]

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10 The function f is defined by

$$\mathrm{f}: x \to \begin{cases} a^2x - ax^2 & \text{for } x \le \frac{1}{2}a, \\ -\frac{1}{2}a^2x + \frac{1}{2}a^3 & \text{for } x > \frac{1}{2}a, \end{cases}$$

where a is a positive constant.

(i) Sketch the graph of $y = \mathrm{f}(x)$, indicating the coordinates of the points where the curve cuts the x-axis and other major features. [2]

(ii) If the domain of f is further restricted to $x \le k$, state with a reason the greatest value of k, in terms of a, for which the function f^{-1} exists. [2]

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(iii) Using the value of k found in part **(ii)**, express the definition of f^{-1} in similar form. State the relationship between the graphs of f and f^{-1}. [4]

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10 [Continued]

The function g is defined by

$$g : x \rightarrow e^x,\ x \leq a^3.$$

(iv) State whether the composite function gf exists, justifying your answer. [1]

(v) Find the range of gf. [1]

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11 A drug ‘Painomore’ has a half-life of 4 hours in the bloodstream, where half-life of a drug is defined to be the time taken for the drug to reduce by 50% in the bloodstream. The drug is formulated to be administered in doses of D milligrams for every 12 hours.

(i) Find the amount of ‘Painomore’ that is left in the bloodstream just before the second dose is administered, giving your answer in terms of D. [1]

(ii) If U_n is the amount of ‘Painomore’ left in the bloodstream after the n^{th} dose, show that $U_n = kD\left(1-\left(\frac{1}{8}\right)^n\right)$ where k is an exact constant to be determined. [3]

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11 [Continued]

(iii) An amount of more than 60 milligrams of 'Painomore' in the bloodstream is considered toxic. Find the largest possible value of D for the drug to be given repeatedly over a long period of time without harming the patient. [2]

For subsequent parts of the question, use $D = 40$.

(iv) Find the least number of doses to be administered to the patient so that the amount of drug in the bloodstream after the last dose is within 4.3 milligrams of 50 milligrams. [3]

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(v) For a patient with a special medical condition, the maximum recommended dosage of ‘Painomore’ is 20 milligrams to be taken once daily.

A doctor advised his patient with the special medical condition to take 4 milligrams of the drug on the first day and increase the intake of the drug by 2 milligrams every subsequent day until the maximum dose of 20 milligrams is reached. Thereafter, the patient is to take 20 milligrams daily over the remaining course of the medication. Find the total amount of drug the patient has taken over the entire course of 14 days. [3]

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12 A player is controlling an avatar in a virtual reality game where points (x, y, z) are defined relative to a fixed point O. Units are measured in metres and we assume that the avatar moves in a straight line from point to point.

Initially the avatar is at a point A with coordinates $(-3, 4, -1)$. The avatar first travels in the direction parallel to $\mathbf{i}-3\mathbf{j}$ towards a surface p_1, with equation $x+y+3=0$. The avatar is supposed to destroy the target at a point B with coordinates $(-1, -2, -1)$ so that it could pass through the surface p_1 and move towards its final destination at a point C.

(i) Verify that the target at B lies on p_1. [1]

(ii) Find the shortest distance from A to p_1, giving your answer in exact form. [3]

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(iii) Find the acute angle the path AB makes with p_1. [2]

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12 **[Continued]**

(iv) While the avatar is moving from A to B, a virtual replica moves from the point A' to B such that the paths AB and $A'B$ are symmetrical about the surface p_1. Find the vector equation of the straight path taken by the virtual replica. [4]

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(v) After destroying the target at B, the avatar passes through the surface p_1 and moves towards C in the direction parallel to $3\mathbf{i}+\mathbf{j}+2\mathbf{k}$. C lies on another surface p_2 with equation in the form of $x+y=k$, where k is a positive constant. Given that the distance between p_1 and p_2 is $35\sqrt{2}$ metres, find the value of k and the coordinates of C. [4]

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