CJC 2019 H2 Mathematics

Prelim Paper 1

1 Differentiate e^{-x^2} with respect to x. Hence find $\int x^3 e^{-x^2}\,dx$. [3]

2 Without using a calculator, solve the inequality $x^2+4x+3<\dfrac{x+3}{2x+1}$. [4]

3 The curve C has the equation $\dfrac{(x-5)^2}{16}+\dfrac{(y-3)^2}{9}=1$.

(i) Sketch C, showing clearly the coordinates of the stationary points and the vertices. [2]

The region R is bounded by curve C, the line $x = 9$ and the x axis.

(ii) Find the volume of the solid generated when region R is rotated completely about the x axis. [3]

(iii) Describe fully a sequence of two transformations which would transform the curve C to the curve $\dfrac{(2x-5)^2}{16}+\dfrac{y^2}{9}=1$. [2]

4 (i) Show that $\dfrac{3r(r+1)+1}{r^3(r+1)^3}=\dfrac{A}{r^3}+\dfrac{B}{(r+1)^3}$, where A and B are to be determined. [2]

(ii) Hence find $\dfrac{7}{(1)^3(2)^3}+\dfrac{19}{(2)^3(3)^3}+\ldots+\dfrac{6n(2n+1)+1}{(2n)^3(2n+1)^3}$. [3]

(iii) Use your answer in part (ii) to find $\sum_{r=1}^{\infty}\left[\dfrac{3r(r+1)+1}{r^3(r+1)^3}+\left(\dfrac{1}{2}\right)^r\right]$. [3]

5 A developer has won the tender to build a stadium. Sunrise Singapore, who would finance the construction, wants to have a capacity of at least 50 000 seats. There is a limited land area to the stadium, and in order for the stadium to have a full sized track and football pitch, the first row of seats can seat 300 people, and every subsequent row has an additional capacity of 20 seats.

(i) What is the least number of rows the stadium must have to meet Sunrise Singapore's requirement? [3]

Assume now that the stadium has been built with 60 rows, with row 1 nearest to the football pitch.

(ii) For the upcoming international football match between Riverloop FC and Gunners FC, tickets are priced starting with \$60 for Category 1, with each subsequent category cheaper by 10%. How much would a seat in the 45^{th} row cost? [1]

Category 1	Rows 1 – 20
Category 2	Rows 21 – 40
Category 3	Rows 41 – 60

There is a total crowd of 51 000 for the football match.

(iii) Assuming that all the tickets from the cheapest category are sold out first before people purchase tickets from the next category, calculate the total revenue collected for the football match. [5]

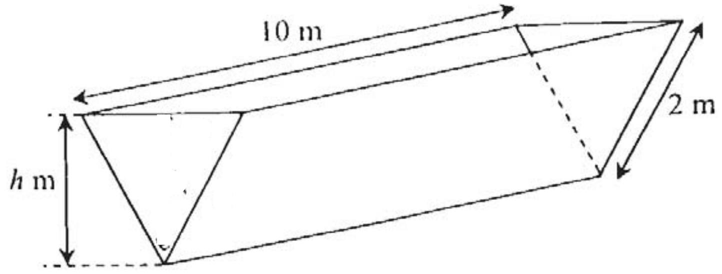
6 (i) Given that $y=\sqrt{e^x \cos x}$, show that $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y\frac{\mathrm{d}y}{\mathrm{d}x} - y^2$. [4]

(ii) Find the Maclaurin series for $y=\sqrt{e^x \cos x}$, up to and including the term in x^3.

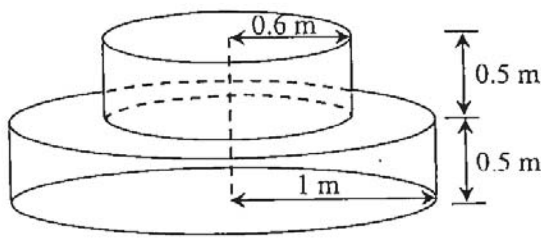
[4]

(iii) Verify the correctness of the expansion found in part (ii) using standard series found in the List of Formulae (MF26). [3]

7 (i) A water trough, shown in the diagram below, in the shape of a triangular prism is used to collect rainwater. The trough consists of two rectangular zinc sheets of negligible thickness, each with fixed dimensions 10 m by 2 m, and two triangular zinc sheets with height h m, as shown in the diagram below. Use differentiation to find the maximum volume of the trough, proving that it is a maximum. [5]



(ii) Water collected in the trough is drained at a rate of 0.001m3 /s into a container consisting of two cylinders, as shown below. The larger cylinder has radius 1 m and height 0.5 m. The smaller cylinder has radius 0.6 m and height 0.5 m. Find the rate at which the depth of water is increasing after 29 minutes. [4]



(iii) Let H be the height of the water level in the container at time t.

Sketch the graph of $\frac{\mathrm{d}H}{\mathrm{d}t}$ against t. [3]

8 **Do not use a calculator in answering this question.**

(a) Showing your working clearly, find the complex number z and w which satisfy the simultaneous equations

$$3z - \mathrm{i}w = 12 + 9\mathrm{i},$$
$$2z^* + 3w = 16 - 23\mathrm{i}.$$

[5]

(b) It is given that z_1 is a root of the equation $z^4+3z^3+4z^2-8=0$, where $z_1=-1+\sqrt{3}\mathrm{i}$.

(i) Express $z^4+3z^3+4z^2-8=0$ as a product of two quadratic factors with real coefficients. [4]

(ii) Given that $\mathrm{e}^{p+\mathrm{i}q}=z_1^5$, determine the exact values of p and q, where $-\pi<q<\pi$. [4]

9 Line l_1 has equation $r=\begin{pmatrix}-1\\8\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\5\end{pmatrix}$ where λ is a real parameter, and plane p has equation $\mathbf{r}\cdot\begin{pmatrix}a\\b\\0\end{pmatrix}=17$. It is given that l_1 lies completely on p and the point Q has coordinates $(-1,-2,3)$.

(i) Show that $a = -1$ and $b = 2$. [3]

(ii) Find the foot of perpendicular from point Q to point P. Hence find the shortest distance between point Q and p in exact form. [4]

Given that the shortest distance between l_1 and the foot of perpendicular of Q on p is $\frac{5}{3}\sqrt{6}$

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(iii) Using the result obtained in part (ii) or otherwise, find the shortest distance between Q and l_1. [2]

The line l_2 is parallel to p, perpendicular to l_1 and passes through P and Q.

(iv) Show that the Cartesian equation of the line l_2 is $\frac{x+1}{2}=y+2=3-z$. [3]

(v) Find the vector equation of line l_3, which is the reflection of l_2 about p. [3]

10 Tumours develop when cells in the body divide and grow at an excessive rate. If the balance of cell growth and death is disturbed, a tumour may form. A medical scientist investigates the change of the tumour size, L mm at time t days of a particular patient using models A and B. For both of the models, it is given that the initial rate of the tumour size is 1 mm per day when the tumour size is 1 mm.

(i) Under Model A, the scientist observes that the patient's turmour is growing at a rate proportional to the square root of its size. At the same time, the tumour is reduced by radiation at a rate proportional to its size. It is further observed that the patient's tumour is decreasing at 2 mm per day when the tumour is 4 mm.

Show that L and t are related by the differential equation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = 3\sqrt{L} - 2L.$$ [2]

(ii) Using the substitution $L = y^2$ where $y > 0$, show that the differential equation in part (i) can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3-2y}{2}.$$

Find y in terms of t and hence find L in terms of t only. [7]

(iii) Under Model B, the scientist suggests that L and t are related by the differential equation

$$\frac{\mathrm{d}^2L}{\mathrm{d}t^2} = \frac{-2t}{(1+t^2)^2}.$$

Find the particular solution of this differential equation. [4]

(iv) Find tumour sizes predicted by Models A and B in the long run. [2]